The Pointwise Difference Operation and the Lattice–theoretical Difference Operation on Fuzzy Sets

Zheng Yalin, Zhang Wenxiu

(Institute of Information and System Science, Faculty of Science, Xi'an Jiaotong University, Xi'an 710049, the Prople's Republic of China)

Abstract

In this paper, the pointwise difference operation and lattice–theoretical difference operation on fuzzy sets are defined, their basic properties are discussed, and the de Morgan dual laws about the two new kinds of difference operations are established.

Key words:fuzzy set, pointwise difference operation, lattice–theoretical difference operation, symetric difference operation, de Morgan dual laws.

1. The pointwise difference operation of fuzzy sets

Via investigate the difference $A-B=\{x\in X|x\in A,\ x\notin B\}$ of two classical sets $A,B\subset X$, it can be discovered that the characteristic function

$$\chi_{A-B}(x)=\begin{cases}1, & x\in A-B,\\ 0, & x\notin A-B,\end{cases}$$

$$=\begin{cases}1, & x\in A, X\notin B,\\ 0, & otherwise\end{cases}$$

of A−B may be expressed equivalently as following:

$$\chi_{A-B}(x)=\begin{cases}1, & A(x)=1, B(x)=0,\\ 0, & otherwise,\end{cases}$$

$$=\begin{cases}A(x)-B(x), & A(x)>B(x),\\ 0, & A(x)\leqslant B(x).\end{cases}$$

The pointwise expression of the difference operation of classical sets may be generalized to the case of fuzzy sets at once.

1.1. Definition. For every pair of fuzzy sets A,B $\in$ F(X), make a new fuzzy set A−B as following: for each x $\in$ X, let

$$(A-B)(x)=\begin{cases}A(x)-B(x), & A(x)>B(x),\\ 0, & A(x)\leqslant B(x).\end{cases} \quad (1)$$

The new fuzzy set A−B is called the *Pointwise difference* of A and B.

It is clear that (1) may be expressed equivalently as

$$(A-B)(x)=\begin{cases}A(x)-B(x), & A(x)\geqslant B(x),\\ 0, & A(x)<B(x),\end{cases} \quad (2)$$

or

$$(A-B)(x)=\begin{cases}A(x)-B(x), & A(x)\geqslant B(x),\\ 0, & A(x)\leqslant B(x).\end{cases} \quad (3)$$

It is easy to see that the pointwise difference χ_x−A of the greatest fuzzy set χ_x and an arbitrary fuzzy set A $\in$ P(X) is just the usual pseudo−complement A′ of A in the theory of fuzzy sets.

1.2. Proposition. For every fuzzy set A $\in$ F(X),

$$A'=\chi_x-A. \quad (4)$$

It is easy to see that χ_Φ is just the zero element about the pointwise difference operation.

1.3 Proposition. For every fuzzy set A $\in$ F(X),

(i) A−A = χ_Φ;

(ii)A−χ_Φ = A.

It is easy to see that de Morgan dual laws associated with the pointwise difference operation must hold.

1.4. Proposition. For all fuzzy sets A,B,C $\in$ F(X),

(i) A−(B$\vee$C) = (A−B)$\wedge$(A−C);

(ii) A−(B$\wedge$C) = (A−B)$\vee$(A−C).

Proof. (i) Let x $\in$ X.

(a) If A(x)$\leqslant$B(x) or A(x)$\leqslant$C(x), then

$$(A-B)(x)\wedge(A-C)(x)=0,$$

that is

$$((A-B)\wedge(A-C))(x)=0.$$

But from $A(x) \leqslant B(x)$ or $A(x) \leqslant C(x)$ we see that

$$A(x) \leqslant B(x) \vee C(x) = (B \vee C)(x),$$

so

$$(A-(B \vee C))(x) = 0.$$

Hence

$$(A-(B \vee C))(x) = ((A-B) \vee (A-C))(x).$$

(b) If $A(x) > B(x)$ and $A(x) > C(x)$, then

$$(A-B)(x) = A(x)-B(x),$$
$$(A-C)(x) = A(x)-C(x),$$

so that

$$\begin{aligned}((A-B) \wedge (A-C))(x) &= (A-B)(x) \wedge (A-C)(x) \\ &= (A(x)-B(x)) \wedge (A(x)-C(x)) \\ &= A(x)-(B(x) \vee C(x)) \\ &= A(x)-(B \vee C)(x).\end{aligned}$$

But from $A(x) > B(x)$ and $A(x) > C(x)$ we see that

$$A(x) > B(x) \vee C(x) = (B \vee C)(x).$$

Hence

$$(A-(B \vee C))(x) = A(x)-(B \vee C)(x).$$

Therefore

$$(A-(B \vee C))(x) = ((A-B) \wedge (A-C))(x).$$

Combine (a) and (b) we abtain that

$$A-(B \vee C) = (A-B) \wedge (A-C).$$

(ii) let $x \in X$.

(a) If $A(x) \leqslant B(x)$ and $A(x) \leqslant C(x)$, then

$$(A-B)(x) = 0 \text{ and } (A-C)(x) = 0.$$

So that

$$(A-B)(x) \vee (A-C)(x) = 0,$$

that is

$$((A-B) \vee (A-C))(x) = 0.$$

But from $A(x) \leqslant B(x)$ and $A(x) \leqslant C(x)$ we see that

$$A(x) \leqslant B(x) \wedge C(x) = (B \wedge C)(x),$$

and then

$$(A-(B \wedge C))(x) = 0.$$

Therefore

$$(A-(B \wedge C))(x) = ((A-B) \vee (A-C))(x).$$

(b) If $A(x) > B(x)$ and $A(x) > C(x)$, then $(A-B)(x) = A(x)-B(x)$ and $(A-C)(x) = A(x)-C(x)$,

so that

$$\begin{aligned}((A-B) \vee (A-C))(x) &= (A-B)(x) \vee (A-C)(x) \\ &= (A(x)-B(x)) \vee (A(x)-C(x)) \\ &= A(x)-(B(x) \wedge C(x)) \\ &= A(x)-(B \wedge C)(x).\end{aligned}$$

But from $A(x) > B(x)$ and $A(x) > C(x)$ we see that

$$A(x) > B(x) \wedge C(x) = (B \wedge C)(x),$$

hence

$$(A-(B \wedge C))(x) = A(x)-(B \wedge C)(x),$$

therefore

$$(A-(B \wedge C))(x) = ((A-B) \vee (A-C))(x).$$

(c) If $A(x) > B(x)$ and $A(x) \leqslant C(x)$,

then

$$(A-B)(x) = A(x)-B(x) \text{ and } (A-C)(x) = 0,$$

so that

$$\begin{aligned}((A-B) \vee (A-C))(x) &= (A-B)(x) \vee (A-C)(x) \\ &= (A(x)-B(x)) \vee 0 \\ &= A(x)-B(x).\end{aligned}$$

But from $B(x) < A(x) \leqslant C(x)$ we see that

$$A(x) > B(x) \wedge C(x) = (B \wedge C)(x).$$

Hence

$$\begin{aligned}(A-(B \wedge C))(x) &= A(x)-(B \wedge C)(x) \\ &= A(x)-(B(x) \wedge C(x))\end{aligned}$$

$$= A(x) - B(x).$$

Therefore

$$(A-(B\wedge C)(x) = ((A-B)\vee(A-C))(x).$$

(d) If $A(x) \leqslant B(x)$ and $A(x) > C(x)$, then one can easily verify that

$$(A-(B\wedge C))(x) = ((A-B)\vee(A-C))(x).$$

Combine (a), (b),(c), and (d), we obtain that

$$A-(B\wedge C) = (A-B)\vee(A-C). \quad \blacksquare$$

The first de Morgan daul law can be generalized to the case of infinite operations.

1.5. Proposition. Take arbitrarily a family of fuzzy sets $\{B_\alpha | \alpha \in D\} \subset F(X)$, then for each fuzzy set $A \in F(X)$,

$$A - (\bigvee_{\alpha\in D} B_\alpha) = \bigwedge_{\alpha\in D} (A - B_\alpha).$$

1.6. Note. Generally,

$$A\wedge(B-C) \neq (A\wedge B)-(A\wedge C).$$

For example, take $A \equiv 0.6$, $B \equiv 0.8$, $C \equiv 0.1$, then for every $x \in X$,

$$\begin{aligned}(A\wedge(B-C))(x) &= A(x)\wedge(B-C)(x)\\ &= A(x)\wedge(B(x)-C(x))\\ &= 0.6\wedge(0.8-0.1)\\ &= 0.6\wedge 0.7\\ &= 0.6.\end{aligned}$$

But

$$(A\wedge B)(x) = A(x)\wedge B(x) = 0.6\wedge 0.8 = 0.6,$$

$$(A\wedge C)(x) = A(x)\wedge C(x) = 0.6\wedge 0.1 = 0.1,$$

$$\begin{aligned}((A\wedge B)-(A\wedge C))(x) &= (A\wedge B)(x)-(A\wedge C)(x)\\ &= 0.6-0.1\\ &= 0.5.\end{aligned}$$

Here,

$$A\wedge(B-C) \neq (A\wedge B)-(A\wedge C).$$

1.7. Definition. For every pair of fuzzy sets $A, B \in F(X)$, we make a new fuzzy set

$$A \oplus B = (A-B) \vee (B-A)$$

and call $A \oplus B$ the *pointwise symetric difference* of A and B.

1.8. Proposition. For arbitrary fuzzy sets $A, B \in F(X)$,

(Ⅰ) $A \oplus A = \chi_\Phi$;

(Ⅱ) $A \oplus \chi_\Phi = A$;

(Ⅲ) $A \oplus \chi_\Phi = A'$;

(Ⅳ) $A \oplus B = B \oplus A$.

2. The lattice–theoretical difference operation of fuzzy sets

The difference

$$A-B = \{x \in X | x \in A, x \notin B\}$$

of classical sets $A, B \subset X$ also can be expressed equivalently as

$$A-B = A \cap B'.$$

Take this lattice–theoretical expression as starting point, we obtain another difference operation of fuzzy sets at once, it is also a generalization of the difference operation of classical sets.

2.1. Definition. For every pair of fuzzy sets $A, B \in F(X)$, we make a new fuzzy set

$$A-B = A \wedge B'$$

and call A–B as the *lattice–theoretical difference* of A and B.

Similarly, the lattice–theoretical difference $\chi_x - A$ of the greatest fuzzy set χ_x and an arbitrary fuzzy set $A \in P(X)$ is just the usual pseudo–complement A' of A in the theory of fuzzy sets.

2.2. Proposition. For every fuzzy set $A \in F(X)$,

$$\chi_x - A = A'.$$

2.3. Proposition. For every fuzzy set $A \in F(X)$,

$$A - \chi_\Phi = A.$$

2.4. Note. But for a fuzzy set $A \in F(X)$, equality

$$A - A = \chi_\Phi$$

needn't hold generally, because equality

$$A\wedge A'=\chi_{\Phi}$$

does not hold generally. Thereby combine the Note and Proposition 1.3.(i), we understand that the two new kinds of difference operation of fuzzy sets defined by us are different, although they are all the generalizations of the difference operation of classical sets.

2.5. Proposition. For every pair of fuzzy sets $A,B\in F(X)$,

(i) $A-B'=B-A'$;

(ii) $A-B=B'-A'$.

Similarly, de Morgan dual laws associated with the lattice–theoretical difference operation must hold yet.

2.6. Proposition. For all fuzzy sets $A,B,C\in F(X)$,

(i) $A-(B\vee C)=(A-B)\wedge(A-C)$;

(ii) $A-(B\wedge C)=(A-B)\vee(A-C)$.

The two de Morgan dual laws can be generalized to the case of infinite operations.

2.7. Proposition. For every family of fuzzy sets $\{B_\alpha|\alpha\in D\}\subset F(X)$ and an arbitrary fuzzy set $A\in F(X)$,

(i) $A-(\bigvee_{\alpha\in D} B_\alpha)=\bigwedge_{\alpha\in D}(A-B_\alpha)$;

(ii) $A-(\bigwedge_{\alpha\in D} B_\alpha)=\bigvee_{\alpha\in D}(A-B_\alpha)$.

2.8. Proposition. For all fuzzy sets $A,B,C\in F(X)$,

(Ⅰ) $(A\wedge B)-(A\wedge C)=(A\wedge B-A)\vee(A\wedge B-C)$;

(Ⅱ) $(A\wedge B)-(A\wedge C)=A\wedge((B-A)\vee(B-C))$;

(Ⅲ) $(A\wedge B)-(A\wedge C)=A\wedge(B-A\wedge C)$;

(Ⅳ) $(A\wedge B)-(A\wedge C)=B\wedge(A-A\wedge C)$.

2.9. Propostion. For all fuzzy sets $A,B,C\in F(X)$,

(Ⅰ) $A\wedge(B-C)=B\wedge(A-C)$;

(Ⅱ) $A\wedge(B-C)=A-(B'\vee C)$;

(Ⅲ) $(A-B)-C=A-(B\vee C)$;

(Ⅳ) $(A-B)-C=(A-C)-B$;

(Ⅴ) $(A-C)-(B-C)=(A-B\vee C)\vee(A\wedge C-C)$;

(Ⅵ) $(A-C)-(B-C)=A\wedge((B'-C)\vee(C-C))$;

(Ⅶ) $(A-C)-(B-C)=(A-B\wedge C')-C$;

(Ⅷ) $(A-C)-(B-C)=(A-(B-C))-C$.

2.10. Definition. For every pair of fuzzy sets $A,B\in F(X)$, we make a new fuzzy set

$$A\oplus B=(A-B)\vee(B-A),$$

and then call $A\oplus B$ the *lattice–theoretical symetric difference* of A and B.

2.11. Proposition. For every pair of fuzzy sets $A,B\in F(X)$,

(i) $A\oplus B=B\oplus A$;

(ii) $A\oplus B=A'\oplus B'$.

2.12. Proposition. For every fuzzy set $A\in F(X)$,

(i) $A\oplus\chi_\Phi=A$;

(ii) $A\oplus\chi_x=A'$.

2.13. Proposition. For all fuzzy sets $A,B,C\in F(X)$,

(Ⅰ) $(A\wedge B)\oplus(A\wedge C)=A\wedge[(B\oplus C)\vee(B-A)\vee(C-A)]$;

(Ⅱ) $(A\vee B)\wedge(A'\vee B')=(A-A)\vee(A\oplus B)\vee(B-B)$;

(Ⅲ) $(A\oplus B)\vee(A\wedge B)=(A-B\wedge B')\vee(B-A\wedge A')$;

(Ⅳ) $(A\oplus B)\vee(A\wedge B)=(A-(B-B))\vee(B-(A-A))$.

References

〔1〕 Zhang Wenxiu, Basic Theory of Fuzzy Mathematics, Xi'an Jiaotong University Press, 1984.

〔2〕 Zhang Wenxiu, Wang Guojun, Liu Wangjin, Fang Jinxuan, An Introduction to Fuzzy Mathematics, Xi'an Jiaotong Unversity Press, 1991.

〔3〕 Zhang Wenxiu, Leung Yee, The Uncertainty Reasoning Principles, Xi'an Jiaotong University Press, 1996.

〔4〕 Zhang Wenxiu, Leung K S, Fuzzy Control and Systems, Xi'an Jiaotong Unversity Press, 1998.