The structure of generated fuzzy algebraic system

Qingde Zhang
Department of Computer, Liaocheng Teachers University, Shandong 252059

Abstract: In this paper, we introduce the concept of generated fuzzy algebraic system and study its structure with the technique of level subset.

Keywords: Algebraic system; Fuzzy algebraic system; generated fuzzy algebraic system.

1. Introduction

The notion of fuzzy sets was introduced by Zadeh[6]. Rosenfeld[7] developed this notion in abstract algebra and introduced the notion of fuzzy subgroup. Since then the study of various fuzzy algebraic concepts has been developing in depth, such as fuzzy rings, fuzzy module, fuzzy field and so on. Papers [1], [2] and [5] sum up the common law and gives the fuzzify method to abstract algebra. Paper [3] and [4] investigated generated fuzzy subgroup and generated fuzzy ring. This paper will discuss the generated fuzzy algebraic systems and give a general fuzzify generated method of various algebraic structure.

2. The concepts and properties of fuzzy algebraic system

Definition 2.1 A algebraic system means a triple S=(X,F,A), where X a nonempty set; F($\neq\varnothing$) a set of operations on X, i.e., for each f∈F, f means a m-ary(m⩾1) operation:

$$f:\ X^m \to X,\quad (x_1,x_2,\ldots,x_m)\,\alpha\ f(x_1,x_2,\ldots,x_m)\in X;$$

A a set of axioms, which the operation of F must follows.

Definition 2.2 Let S=(X,F,A) be a algebraic system and $\varnothing\neq Y\subset X$. If for any m-ary operation f∈F and $x_1,x_2,\ldots,x_m\in$X, $f(x_1,x_2,\ldots,x_m)\in Y$, then T=(Y,F,A) is called subalgebraic system. of S.

Definition 2.3 Let S=(X,F,A) be a algebraic system, μ a fuzzy subset of X. If for any m-ary operation f∈F and $x_1,x_2,\ldots,x_m\in$X,

$$\mu(f(x_1,x_2,\ldots,x_m))\geq\min(\mu(x_1),\mu(x_2),\ldots,\mu(x_m)),$$

then S=(X,F,A) is called a fuzzy algebraic system of S.

It is easy to see Definition2.3 generalized the notions such as fuzzy group, fuzzy ring, fuzzy field, fuzzy module and so on.

Theorem2.1 Let S=(X,F,A) be a algebraic system, μ a fuzzy subset of X. Then the following statements are equivalent:

(1) μ is a fuzzy algebraic system of S,

(2) μ_λ is a crisp subalgebraic system of S,

(3) $\mu_{\underset{\cdot}{\lambda}}$ is a crisp subalgebraic system of S.

Proof. (1)⇒(2) Let μ be a fuzzy algebraic system of S, for any m-ary operation f ∈ F and $x_1, x_2, \ldots, x_m \in \mu_\lambda (\lambda \in [0,1])$, we have $\mu(x_i) \geq \lambda (i = 1, \ldots, m)$,because μ is fuzzy algebraic system, so $\mu(f(x_1, x_2, \ldots, x_m)) \geq \min(\mu(x_1), \mu(x_2), \ldots, \mu(x_m)) \geq \lambda$, this means $f(x_1, x_2, \ldots, x_m) \in \mu_\lambda$, by Definiton2.2 μ_λ is a subalgebraic system.

(2)⇒(1) If μ_λ is a subalgebraic system for each $\lambda \in [0,1]$, then for any m-ary operation f ∈ F and $x_1, x_2, \ldots, x_m \in \mu_\lambda (\lambda \in [0,1])$, we have $f(x_1, x_2, \ldots, x_m) \in \mu_\lambda$. For any $x_1, x_2, \ldots, x_m \in X$, suppose $\min_{1 \leq i \leq m} \mu(x_i) = \lambda$, then $x_i \in \mu_\lambda$ (I=1,2,…,m), thus $f(x_1, x_2, \ldots, x_m) \in \mu_\lambda$, and then

$$\mu(f(x_1, x_2, \ldots, x_m)) \geq \lambda = \min(\mu(x_1), \mu(x_2), \ldots, \mu(x_m)),$$

by Definition2.3 (1) hold.

(1)⇒(3) Similar to (1)⇒(2).

(3)⇒(1) For any m-ary operation f ∈F and $x_1, x_2, \ldots, x_m \in X$, let $\min_{1 \leq i \leq m} \mu(x_i) = \lambda$, then $x_i \in \mu_{\underset{\cdot}{\lambda-\varepsilon}}$ for any $0 < \varepsilon < \lambda$, so $f(x_1, x_2, \ldots, x_m) \in \mu_{\underset{\cdot}{\lambda-\varepsilon}}$(for $\mu_{\underset{\cdot}{\lambda-\varepsilon}}$ is crisp subalgebraic system) and then $\mu(f(x_1, x_2, \ldots, x_m)) > \lambda - \varepsilon$, by the arbitrary of ε we have $\mu(f(x_1, x_2, \ldots, x_m)) \geq \lambda = \min(\mu(x_1), \mu(x_2), \ldots, \mu(x_m))$, according to Definition2.3 μ a fuzzy algebraic system. □

Definition2.4 Let S=(X,F,A) be a algebraic system, Y⊂X. The minimal subalgebraic system of S, which contains Y, is called the generated algebraic system of Y, denoted by <Y>.

3. The generated fuzzy algebraic system and its structure

Definition3.1 Let S=(X,F,A) be a algebraic system, μ a fuzzy set of X. The minimal fuzzy algebraic system of S, which contains μ , is called the generated fuzzy algebraic system of μ, denoted by $<\mu>$.

Obviously, $<\mu>$ is determined by μ.

Theorem3.1 Let S=(X,F,A) be a algebraic system, μ a fuzzy set of X and Im $\mu<\infty$. Suppose $\mu(x)=\{t_1,t_2,...,t_m\}$, $0\le t_1<t_2<...<t_m\le 1$, we define the fuzzy set μ^* of X as follows:

$$\mu^*(x)=\begin{cases} t_m & x\in<\mu_{t_m}> \\ t_i & x\in<\mu_{t_i}>\setminus<\mu_{t_{i+1}}> \ (1\le i\le m) \end{cases},$$

then $<\mu>=\mu^*$.

Proof. From $0\le t_1<t_2<...<t_m\le 1$ we know, $\mu_{t_1}\supset\mu_{t_2}\supset...\supset\mu_{t_m}$, and then $<\mu_{t_1}>\supset<\mu_{t_2}>\supset...\supset<\mu_{t_m}>$. It is not difficult to see: for any $x\in X$, either $x\in<\mu_{t_m}>$ or there exist unique $i(1\le i\le m)$ such that $x\in<\mu_{t_i}>\setminus<\mu_{t_{i+1}}>$. This means the definition of μ^* is reasonable.

Next we prove $<\mu>=\mu^*$.

(1) μ^* is a fuzzy algebraic system of S.

For any m-ary operation f $\in$ F and $x_1,x_2,...,x_m\in X$, suppose $\mu^*(x_i)=t_{r(i)}$ and $t_{r(1)}\le t_{r(2)}\le...\le t_{r(m)}$ (where $r(i)\in \mathrm{Im}(\mu), i=1,2,...,m$), then $x_i\in<\mu_{t_{r(i)}}>\subset<\mu_{t_{r(1)}}>$. But $<\mu_{t_{r(1)}}>$ is a crisp subalgebraic system of S, so $f(x_1,x_2,...,x_m)\in<\mu_{t_{r(1)}}>$. By the definition of μ^*, $\mu^*(f(x_1,x_2,...,x_m))\ge t_{r(1)}=\{\mu^*(x_1),...,\mu^*(x_m)\}$, this show that μ^* is a fuzzy algebraic system of S.

(2) $\mu\subset\mu^*$.

If $\mu(x)=t_m$, then $x\in<\mu_{t_m}>$, and then $\mu^*(x)=\mu(x)=t_m$; If $\mu(x)=t_i(1\le i<m)$, then $x\in<\mu_{t_j}>(1\le j\le i)$, and then $x\notin<\mu_{t_{j-1}}>\setminus<\mu_{t_j}>$, thus exist some $k\ge i$ such that $x\in<\mu_{t_k}>\setminus<\mu_{t_{k-1}}>$, that is $\mu^*(x)=t_k\ge t_i=\mu(x)$.

(3) μ^* is minimal

Let η be any fuzzy algebraic system of S and $\mu\subset\eta$. If $\mu(x)=t_m$, then $x\in<\mu_{t_m}>$, because $\mu_{t_m}\subset\eta_{t_m}$ and η_{t_m} is a crisp subalgebraic system, we obtain $<\mu_{t_m}>\subset\eta_{t_m}$, and then $x\in\eta_{t_m}$, $\eta(x)\ge t_m=\mu^*(x)$; If $\mu(x)=t_i(1\le i<m)$, then there exist $k\ge i$ such that $x\in<\mu_{t_k}>\setminus<\mu_{t_{k-1}}>$, so $x\in<\mu_{t_k}>\subset\eta_{t_k}$, $\eta(x)\ge t_k=\mu^*(x)$.

Sum up (1),(2) and (3) above, we have $<\mu>=\mu^*$. □

Theorem3.2 Let S=(X,F,A) be a algebraic system, μ a fuzzy set of X.

For any $x \in X$, let $J_x = \{t | t \in \mu(x)\ and\ x \in <\mu_t>\}$. We define μ^* as follows:

$$\mu^*(x) = \sup J_x,$$

then $<\mu> = \mu^*$.

Proof. For any x∈X, suppose $\mu(x) = t$, then $t \in \mu(x)$ and $x \in \mu_t \subset <\mu_t>$, so $t \in J_x$. That is $J_x \neq \varnothing$, and then μ^* is reasonable.

(1) μ^* Is a fuzzy algebraic system of S.

For any m-ary operation f∈F and $x_1, x_2, ..., x_m \in X$, let $\min\limits_{1 \le i \le m} \mu^*(x_i) = t$, then $\mu^*(x_i) > t - \varepsilon$ for any $0 < \varepsilon < t$, so $x_i \in <\mu_{t-\varepsilon}>$, but $<\mu_{t-\varepsilon}>$ is a crisp subalgebraic of S, we have $f(x_1, x_2, ..., x_m) \in <\mu_{t-\varepsilon}>$, and then $\mu^*(f(x_1, x_2, ..., x_m)) \ge t - \varepsilon$. By the arbitrary of ε we obtain

$$\mu^*(f(x_1, x_2, ..., x_m)) \ge t = \min\{\mu^*(x_1), \mu^*(x_2), ..., \mu^*(x_m)\}.$$

This shows that μ^* is a fuzzy algebraic system of S.

(2) $\mu \subset \mu^*$

For any $x \in X$, let $\mu(x) = t$, then $x \in \mu_t \subset <\mu_t>$, thus $\mu^*(x) \ge t = \mu(x)$.

(3) μ^* is minimal

Let η be any fuzzy algebraic system of S, which contains μ. For any $x \in X$ and $t \in J_x$, $x \in <\mu_t> \subset \eta_t$, so $\eta(x) \ge t$, and then $\eta(x) \ge \sup J_x = \mu^*(x)$. That is $\mu^* \subset \eta$.

Sum up above we knows μ^* is generated fuzzy algebraic system of μ.

□

Next, we give another way to construct the generated fuzzy algebraic system . For this we give a lemma first,

Lemma3.1 Let S=(X,F,A) be algebraic system and μ a fuzzy set of X, then

$$<\mu>_{\underset{\cdot}{t}} = <\mu_{\underset{\cdot}{t}}> \subset <\mu_t> \subset <\mu>_t .$$

Proof. It is no hamper to assume $<\mu>_{\underset{\cdot}{t}} \neq \varnothing$. For any $x \in <\mu>_{\underset{\cdot}{t}}$, from Theorem3.2 we have $<\mu>(x) = \sup J_x$, so there exists $s \in J_x$ such that $s > t$, therefore $x \in <\mu_s> \subset <\mu_{\underset{\cdot}{t}}>$, this shows $<\mu>_{\underset{\cdot}{t}} \subset <\mu_{\underset{\cdot}{t}}>$. Conversely, let $x \in \mu_{\underset{\cdot}{t}}$, then $\mu(x) = \alpha > t$, $x \in <\mu_\alpha>$, so $\alpha \in J_x$, $<\mu>(x) \ge \alpha > t$, $x \in <\mu>_{\underset{\cdot}{t}}$, and then $\mu_{\underset{\cdot}{t}} \subset <\mu>_{\underset{\cdot}{t}}$, therefore, we have $<\mu_{\underset{\cdot}{t}}> \subset <\mu>_{\underset{\cdot}{t}}$, so $<\mu_{\underset{\cdot}{t}}> = <\mu>_{\underset{\cdot}{t}}$;

Similarly we can prove $<\mu_t>\subset<\mu>_t$; Obviously $<\mu_{\underset{\cdot}{t}}>\subset<\mu_t>$. □

Theorem3.3 Let S=(X,F,A) be a algebraic system, μ a fuzzy set of X and $\mu\neq 0$, then

$$<\mu>=\bigcup_{0\le t\le 1} t<\mu_{\underset{\cdot}{t}}>=\bigcup_{0\le t\le 1} t<\mu_t>$$

Proof. By Lemma3.1 and decomposition theorem of fuzzy set [6] we have

$$<\mu>=\bigcup_{0\le t\le 1} t<\mu>_{\underset{\cdot}{t}}=\bigcup_{0\le t\le 1} t<\mu>_t$$

and

$$<\mu>=\bigcup_{0\le t\le 1} t<\mu>_t\supset\bigcup_{0\le t\le 1} t<\mu_t>\supset\bigcup_{0\le t\le 1} t<\mu_{\underset{\cdot}{t}}>=\bigcup_{0\le t\le 1} t<\mu>_{\underset{\cdot}{t}}=<\mu>.$$

This completes the proof. □

Reference

[1] Liu Wenbin, A unified fuzzy method and fuzzify to algebraic system, Fuzzy System and Mathematics (in Chinese),3(1989).

[2] Qingde Zhang, Fuzzy algebraic system about t-norm, Journal of Jilin Teachers College (in Chinese), 4(1994).

[3] Qingde Zhang, Generated fuzzy subgroups and generating system of conjugate fuzzy subgroups, BUSEFAL, $n^0$62 (1995).

[4] Fang Jinxuan and Chen Jinguo, The structure of generated fuzzy ring, J.Nanjing Teachers University (in Chinese), 1(1994).

[5] Lutu, Normalized homomorphism and isomorphism of fuzzy algebraic systems, Journal of Jilin Teachers College (in Chinese), 2(1991).

[6] Luo Chengzhong, An introduction to fuzzy sets, Beijing Teachers University Publishing House, Beijing, 1989.

[7] Rosenfeld, Fuzzy groups, J. Math. Anal. Appl. 35(1971).