Conjugacy classes of fuzzy implications *

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Many authors describe similarity relations between binary operations in the unit interval (e.g. characterization of triangular norms in [10], Chapter 5 or characterization of continuous fuzzy implications in [3], Chapter 1). We consider in details a similarity of fuzzy implications.

Definition 1 (cf. [3]). A function $I\colon [0,1]^2 \to [0,1]$ is called *fuzzy implication* if it is monotonic with respect to both variables (separately) and fulfils the binary implication truth-table:

$$I(0,0) = I(0,1) = I(1,1) = 1, \qquad I(1,0) = 0. \tag{1}$$

Set of all fuzzy implications is denoted by FI.

Example 1. The most important multivalued implications (cf. [2]) fulfils the above definition:

$$I_{\mathrm{LK}}(x,y) = \min(1-x+y,1) \qquad \text{(Łukasiewicz [7])}$$

$$I_{RC}(x,y) = 1-x+xy \qquad \text{(Reichenbach [8])}$$

$$I_{GD}(x,y) = \begin{cases} 1, & x \leqslant y \\ y, & x > y \end{cases} \qquad \text{(Gödel [5])}$$

$$I_{DN}(x,y) = \max(1-x,y) \qquad \text{(Dienes [1])}$$

$$I_{GG}(x,y) = \begin{cases} 1, & x \leqslant y \\ \frac{y}{x}, & x > y \end{cases} \qquad \text{(Goguen [4])}$$

$$I_{RS}(x,y) = \begin{cases} 1, & x \leqslant y \\ 0, & x > y \end{cases} \qquad \text{(Rescher [9])}$$

for $x, y \in [0,1]$.

Definition 2 (cf. [6], Chapter 8). Fuzzy implications $I, J \in FI$ are conjugate if there exists a bijection $\varphi\colon [0,1] \to [0,1]$ such that $J = I^*_\varphi$, where

$$I^*(x,y) = I^*_\varphi(x,y) = \varphi^{-1}(I(\varphi(x),\varphi(y))), \qquad \text{for } x,y \in [0,1]. \tag{2}$$

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Theorem 1. *Let $\varphi\colon [0,1] \to [0,1]$ be a bijection. φ is increasing iff*

$$\underset{I\in FI}{\forall} \left(I^*_\varphi \in FI\right). \tag{3}$$

Definition 3. Let Φ denote a family of increasing bijections $\varphi\colon [0,1] \to [0,1]$ and $I, J \in FI$. Fuzzy implication J is Φ-conjugate with I if

$$\underset{\varphi\in\Phi}{\exists} \left(J = I^*_\varphi\right). \tag{4}$$

Theorem 2. *Relation (4) is an equivalence iff $(\Phi, \circ)$ is a group of bijections (with composition operation).*

1 Partial ordering

In FI we can consider partial order induced from $[0,1]$:

$$I \leqslant J \iff \underset{x,y\in[0,1]}{\forall} \left(I(x,y) \leqslant J(x,y)\right). \tag{5}$$

Theorem 3. *Let $I, J \in FI$, $\varphi \in \Phi$. Then*

$$I \leqslant J \iff I^* \leqslant J^*, \tag{6}$$

$$\max(I,J)^* = \max(I^*, J^*), \qquad \min(I,J)^* = \min(I^*, J^*). \tag{7}$$

Theorem 4. *Let $I, J \in FI$. If there exist $\varphi, \psi \in \Phi$ such that I^*_φ is Φ-conjugate with J^*_ψ, then J is Φ-conjugate with I^*_χ, where $\chi = \psi \circ \varphi^{-1}$. Particularly*

$$J^*_\psi \leqslant I^*_\varphi \iff J \leqslant I^*_\chi, \tag{8}$$

$$J^*_\psi = I^*_\varphi \iff J = I^*_\chi. \tag{9}$$

2 Implication classes

Using operation (2) for fuzzy implications listed in Example 1 we obtain

$$I^*_{\mathrm{LK}}(x,y) = \min(\varphi^{-1}(1-\varphi(x)+\varphi(y)), 1), \tag{10}$$

$$I^*_{RC}(x,y) = \varphi^{-1}(1-\varphi(x)+\varphi(x)\varphi(y)), \tag{11}$$

$$I^*_{DN}(x,y) = \max(\varphi^{-1}(1-\varphi(x)), y), \tag{12}$$

$$I^*_{GG}(x,y) = \begin{cases} 1, & x \leqslant y \\ \varphi^{-1}\left(\frac{\varphi(y)}{\varphi(x)}\right), & x > y \end{cases}, \tag{13}$$

for $x, y \in [0,1]$. Moreover $I^*_{GD} = I_{GD}$, $I^*_{RS} = I_{RS}$ (one element conjugacy classes). In general conjugacy classes can be indexed by elements of the group Φ.

Theorem 5. *All implications from formulas (10) and (11) are different.*

Theorem 6. *Implication (12) reduces to I_{DN} iff*

$$\varphi(x) = \begin{cases} h(x), & x \in [0, \frac{1}{2}] \\ 1 - h(1-x), & x \in [\frac{1}{2}, 1] \end{cases},$$

where $h\colon [0, \frac{1}{2}] \to [0, \frac{1}{2}]$ is an increasing bijection.

Theorem 7. *Implications (13) reduces to I_{GG} iff*

$$\underset{\alpha>0}{\exists} (\varphi(x) = x^\alpha,\ x \in [0,1]).$$

3 Bounds of conjugacy classes

Sequences of fuzzy implications can be convergent as sequences of real functions.

Lemma 1. *If (I_n) is a convergent sequences of fuzzy implications then its limit is also a fuzzy implication.*

Theorem 8. *Conjugacy classes of fuzzy implications have greatest lower bounds and least upper bounds. Particularly for fuzzy implications (10)-(13)*

$$\begin{aligned} &\sup I^*_{LK} = \sup I^*_{RC} = \sup I^*_{DN} = J_1, && \sup I^*_{GG} = J_2, \\ &\inf I^*_{LK} = \inf I^*_{GG} = I_{GD}, && \inf I^*_{RC} = \inf I^*_{DN} = J_3, \end{aligned}$$

where

$$J_1(x,y) = \begin{cases} 1, & x < 1 \\ y, & x = 1 \end{cases}, \qquad J_3(x,y) = \begin{cases} 1, & x = 0 \\ y, & x > 0 \end{cases},$$

$$J_2(x,y) = \begin{cases} 1, & x < 1 \wedge y > 0 \vee x = 0 \\ y, & x = 1 \\ 0, & x > 0 \wedge y = 0 \end{cases}.$$

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