Characterization of Dienes implication *

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Our main goal in this paper is to present characterization of implications which are similar to Dienes implication. Our investigations are inspired by the paper of Smets, Magrez [7] where they proved the characterization of implications similar to Łukasiewicz implication. We use here the notation presented by Fodor, Roubens [3].

Definition 1. Any function $I\colon [0,1]^2 \to [0,1]$ is called *fuzzy implication* if it fulfils the following conditions ($x, y, z \in [0,1]$):

I1. $x \leqslant z \Rightarrow I(x,y) \geqslant I(z,y)$,

I2. $y \leqslant z \Rightarrow I(x,y) \leqslant I(x,z)$,

I3. $I(0,y) = 1$,

I4. $I(x,1) = 1$,

I5. $I(1,0) = 0$.

Set of all fuzzy implications will be denoted by FI and set of all continuous fuzzy implications is denoted by CFI.

Example 1. We list here four implication functions completed e.g. by Fodor, Roubens [3]. All of them belong to FI.

$$I_{\mathrm{LK}}(x,y) = \min(1-x+y,1) \qquad \text{(Łukasiewicz [5])}$$

$$I_{GD}(x,y) = \begin{cases} 1, & \text{if } x \leqslant y \\ y, & \text{if } x > y \end{cases} \qquad \text{(Gödel [4])}$$

$$I_{DN}(x,y) = \max(1-x,y) \qquad \text{(Dienes [2])}$$

$$I_{RS}(x,y) = \begin{cases} 1, & \text{if } x \leqslant y \\ 0, & \text{if } x > y \end{cases} \qquad \text{(Rescher [6])}$$

for $x, y \in [0,1]$.

Definition 2. Let $\varphi\colon [0,1] \to [0,1]$ be an increasing bijection, $I \in FI$. We say that the function

$$I^*(x,y) = I^*_\varphi(x,y) = \varphi^{-1}(I(\varphi(x),\varphi(y))), \qquad x,y \in [0,1] \tag{1}$$

is *φ-conjugate* to I. Implication $I \in FI$ is called *selfconjugate* if $I^*_\varphi = I$ for all φ.

*Paper submitted to 6^{th} Fuzzy Days, Dortmund, May, 1999.

Theorem 1. *Let* $\varphi\colon [0,1] \to [0,1]$ *be an increasing bijection. For any* $I \in FI$ ($I \in CFI$)

$$I^*_\varphi \in FI \ \ (I^*_\varphi \in CFI). \tag{2}$$

Example 2. For implications from Example 1 we have

$$I^*_{RS} = I_{RS}, \qquad I^*_{GD} = I_{GD},$$

so this implications are selfconjugate. For next two implications we get new fuzzy implications:

$$I^*_{\mathrm{LK}}(x,y) = \min(\varphi^{-1}(1-\varphi(x)+\varphi(y)),1), \qquad x,y \in [0,1], \tag{3}$$

$$I^*_{DN}(x,y) = \max(\varphi^{-1}(1-\varphi(x)),y), \qquad x,y \in [0,1]. \tag{4}$$

Theorem 2 (Smets, Magrez, [7]). *Function* $I \in CFI$ *satisfies*

(i) $I(x, I(y,z)) = I(y, I(x,z))$, *for all* $x,y,z \in [0,1]$,

(ii) $x \leqslant y \iff I(x,y) = 1$, *for all* $x,y \in [0,1]$

iff there exists an increasing bijection $\varphi\colon [0,1] \to [0,1]$ *such that* $I = I^*_{LK}$.

Definition 3. Any function $n\colon [0,1] \to [0,1]$ is called *strong negation* if it fulfils the following conditions:

(i) $n(0) = 1, \quad n(1) = 0$,

(ii) n is strictly increasing,

(iii) n is continuous,

(iv) $n(n(x)) = x$, for all $x \in [0,1]$.

Theorem 3 (cf. [8]). *A function* $n\colon [0,1] \to [0,1]$ *is a strong negation iff there exists an increasing bijection* $\varphi\colon [0,1] \to [0,1]$ *such that*

$$n(x) = \varphi^{-1}(1-\varphi(x)), \qquad x \in [0,1]. \tag{5}$$

Theorem 4 (cf. [1]). *Function* $S\colon [0,1] \to [0,1]$ *satisfies*

(i) S *is increasing with respect to both variables,*

(ii) $S(x,0) = S(0,x) = x$, *for all* $x \in [0,1]$,

(iii) $S(x,x) = x$, *for all* $x \in [0,1]$

iff $S = \max$.

Lemma 1. *If* $I \in CFI$ *satisfies*

$$I(I(x,0),0) = x, \qquad \textit{for all } x \in [0,1],$$

then function $n\colon [0,1] \to [0,1]$ *defined by* $n(x) = I(x,0)$ *is a strong negation.*

Theorem 5. *Function* $I \in CFI$ *satisfies*

(i) $I(I(x,0),0) = x$, *for all* $x \in [0,1]$,

(ii) $I(I(x,0),x) = x$, *for all* $x \in [0,1]$,

(iii) $I(1,x) = x$, *for all* $x \in [0,1]$

iff there exists an increasing bijection $\varphi\colon [0,1] \to [0,1]$ *such that* $I = I^*_{DN}$.

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