① On the Redefinition of Fuzzy mapping

Chengyi Zhang

(Department of Mathematics, Zhumadian Education College, Henan 463000 China)

Pingan Dang

(Department of Mathematics, Zhumadian Teacher's College, Henan, China)

Abstract In this paper, the fuzzy mapping is redefined, it is reasonabler and make the definitions in [1]and[4]as a special case. In addition, the properties of fuzzy mapping and fuzzy cardinal number are discussed.

Keywords Fuzzy mappings; Fuzzy points, Fuzzy cardinal numbers.

In [1,2,3], Li discussed the fuzzy relations between two fuzzy sets first, based on this definition, a reasonable definition for the mappings of fuzzy sets was given. In [4,5], Author characterized the mapping of fuzzy sets by fuzzy points and studied fuzzy cardional number. But, the fuzzy mappings in [1]and [4] demand both the fuzzy sets are equal—high. It will restrict greatly the application of this fuzzy mappings. In this paper, the fuzzy mapping is redefined, so it is reasonable and make the definitions in [1]and[4]as a special case. In addition, the properties of fuzzy mapping and fuzzy cardinal number are discussed.

1. Introduction

Let X be an ordinary set and $\mathscr{F}(X)$ be the sets of all fuzzy sets in X. $\mathrm{Im}(A)=\{A(x)\mid x\in X\}$ is called the image of A and $h(A)=\vee\{\lambda\mid\lambda\in \mathrm{Im}(A)\}$ is called the highness of A. If $A,B\in\mathscr{F}(X)$, $h(A)=h(B)$, *then we call A and B are equal—high.*

We call x_λ is a fuzzy point in X, $x_\lambda=y_\mu$ iff $x=y$ and $\lambda=\mu$. If $A\in\mathscr{F}(X)$, $A(x)\geqslant\lambda$ denoted $x_\lambda\in A$. When $A(x)=\lambda$, $x_{A(x)}$ is called a principal element (shell—point) of A.

Obviously, $A=\bigcup\{x_{A(x)}\mid x\in X\}$.

In[1], Li had given the mapping between two fuzzy sets.

Definition1. 1[1], Let $A\in\mathscr{F}(X)$, $B\in\mathscr{F}(Y)$, *a fuzzy relation $\underset{\sim}{f}\subset A\times B$ is called a fuzzy mapping from A to B, if* $\forall\lambda\in[0,1]$, *f_λ is a mapping from A_λ to B_λ.*

In [4], Zhang characterized this fuzzy mapping by fuzzy points.

*Definition*1. 2[4]*Let $A\in\mathscr{F}(X)$, $B\in\mathscr{F}(Y)$. If $\forall\, a_{A(a)}\in A$, according the regulation $\underset{\sim}{f}$, exists an unique $b_{B(b)}\in B$, $(B(b)\geqslant A(a))$, then $\underset{\sim}{f}$ is called the fuzzy mapping from A to B.*

In definition 1.1, *A and B are demanded equal—high, i.e. $h(A)=h(B)$; In definition* 1.2, *A and B are demanded to satisfy $h(A)\leqslant h(B)$ for the equivalence of def*1.1 *and def*1.2. *Thus, the application of this mapping is restricted greatly. For example, Let $A\in\mathscr{F}(X)$, $Im(A)=[0,1]$, and $B\in\mathscr{F}(Y)$, $Im(B)=[0,\frac{1}{2}]$, We could not define the fuzzy mapping from A toB.*

2. Redefinition of Fuzzy mapping.

Definition 2.1 Let $A\in\mathscr{F}(X)$, $B\in\mathscr{F}(Y)$, *f be a mapping from $\underset{\cdot}{A}_\circ$ to $\underset{\cdot}{B}_\circ$ and φ be an order—pre-*

① Project supported by the Natural Science Foundation of Shandong.

serving mapping from $[0,h(A)]$ *to* $[o,h(B)]$*, i.e.* $\forall\ \lambda,\mu\in[0,h(A)],\lambda\leqslant\mu$ *iff* $\varphi(\lambda)\leqslant\varphi(\mu)$*. We call* $\underset{\sim}{f}=<f,\varphi>:A\to B,\underset{\sim}{f}(x_\lambda)=(f(x))_{\varphi(\lambda)}$ *is a fuzzy mapping from A to B.*

Denoting dom $\underset{\sim}{f}=A$ *and ran*$\underset{\sim}{f}=\{b_\mu|\underset{\sim}{f}(a_\lambda)=b_\mu,a_\lambda\in A\}=\underset{\sim}{f}(A)$*, It's clear ,* $\underset{\sim}{f}(A)$ *is a fuzzy subset of B.*

Denote: Def 2.1 contains the definition of ordinary mappings.

Let $\underset{\sim}{f}:A\to B$ *be a fuzzy mapping ,the following conclutions are apparent:*

(1) $\underset{\sim}{f}$ *is an order—preserving mapping:* $\forall\ x_\mu\leqslant x_\lambda,\underset{\sim}{f}(x_\mu)\leqslant\underset{\sim}{f}(x_\lambda)$;

(2) *If* $a_{A(a)}$ *is a shell—point of A, then* $\underset{\sim}{f}(a_{A(a)})$ *is a shell—point of* $\underset{\sim}{f}(A)$.

Definition 2.2 Suppose $\underset{\sim}{f}=<f,\varphi>:A\to B$ is a fuzzy mapping. $\underset{\sim}{f}_t=<f,\varphi>_t=<f_t,\varphi>:A_t\to B_{\varphi(t)}$ is called the t—level mapping of $\underset{\sim}{f}$. where $f_t=f|_{A_t}$, $\underset{\sim}{f}_t(a)=(\underset{\sim}{f}(a_{A(a)}))_{\underset{\cdot}{\circ}}=f(a),a\in A_t$.

Obviously, $\underset{\sim}{f}=\bigvee_{t\in[0,h(A)]}<\underset{\sim}{f}_t,\varphi>$, $\underset{\sim}{f}(a_{A(a)})=\bigvee_{\lambda\in[0,h(A)]}\{b_{\varphi(t)}|f_\lambda(a)=b\in B_{\varphi(t)}\}$.

Theorem 2.1 Let $A\in\mathscr{F}(X)$. $B\in\mathscr{F}(Y)$, then $\underset{\sim}{f}=<f,\varphi>:A\to B$ is a fuzzy mapping iff $\forall\ \lambda\in[0,h(A)]$, $\underset{\sim}{f}_\lambda$ is a mapping from A_λ to $B_{\varphi(\lambda)}$。

Proof: (Necessity) Suppose $\lambda\in[0,h(A)],A_\lambda=\{a|A(a)\geqslant\lambda\}$. Let $a\in A_\lambda$, then $\underset{\sim}{f}(a_{A(a)})=f(a)_{\varphi(A(a))}\in B_{\varphi(A(a))}$, since $A(a)\geqslant\lambda$, thus $\varphi(A(a))\geqslant\varphi(\lambda)$, $\underset{\sim}{f}_\lambda(a)\in B_{\varphi(\lambda)}$, $\underset{\sim}{f}_\lambda$ is a mapping from A_λ to $B_{\varphi(\lambda)}$.

(Sufficiency) If $\forall\ \lambda\in[0,h(A)]$, $\underset{\sim}{f}_\lambda$ is a mapping from A_λ to $B_{\varphi(\lambda)}$, when $\lambda,\mu\in[0,h(A)]$ and $\lambda\leqslant\mu$, $\underset{\sim}{f}_\mu=f|_{A_\mu}=\underset{\sim}{f}_\lambda|_{A_\mu}$, so $B_{\varphi(\mu)}\in B_{\varphi(\lambda)}$, thus we have $\varphi(\mu)\leqslant\varphi(\lambda)$, φ is a order—preserving mapping. Let $f=\underset{\sim}{f}_{\underset{\cdot}{\circ}}$, then a fuzzy mapping from A to B $\underset{\sim}{f}=<f,\varphi>$ can be defined, such that $\forall\ a_\lambda\in A$, $\underset{\sim}{f}(a_\lambda)=(f(a))_{\varphi(\lambda)}$. especially, $\underset{\sim}{f}(a_{A(a)})=\bigvee_{t\in[0,h(A)]}\{f(a)_{\varphi(t)}|\underset{\sim}{f}_t(a)\in B_{\varphi(t)}\}=[f(a)]_{\varphi(A(a))}$.

Denote: If $[0,h(A)]=[0,h(B)]=[0,1]$ and φ is identical mapping, $\underset{\sim}{f}$ is the fuzzy mapping in [1].

Definition 2.3 Let $\underset{\sim}{f}=<f,\varphi>:A\to B$ be a fuzzy mapping, then

(1) $\underset{\sim}{f}$ is injective if $\underset{\sim}{f}(x_\lambda)=\underset{\sim}{f}(y_\mu)$ implies $x_\lambda=y_\mu$;

(2) $\underset{\sim}{f}$ is surjective if $\forall\ b_\lambda\in B,\exists\ a_\lambda\in A$, such that $\underset{\sim}{f}(a_\lambda)=b_\mu$;

(3) $\underset{\sim}{f}$ is bijective if $\underset{\sim}{f}$ is injective and surjective.

Theorem 2.2 Let $\underset{\sim}{f}:A\to B$ be a fuzzy mapping ,then

(1) $\underset{\sim}{f}$ is injective iff φ so is and $\forall\ \lambda\in[0,h(A)]$, $\underset{\sim}{f}_\lambda:A_\lambda\to B_{\varphi(\lambda)}$ is injective;

(2) $\underset{\sim}{f}$ is surjective iff φ so is and $\forall\ \lambda\in[0,h(A)]$, $\underset{\sim}{f}_\lambda:A_\lambda\to B_{\varphi(\lambda)}$ is surjective.

(3) $\underset{\sim}{f}$ is bijective iff φ so is and $\forall\ \lambda\in[0,h(A)]$, $\underset{\sim}{f}_\lambda:A_\lambda\to B_{\varphi(\lambda)}$ is bijective.

Noticing the following facts: If $\underset{\sim}{f}:A\to B$ is a fuzzy mapping,

then: (1) $\underset{\sim}{f}_\lambda$ is injective implies $\underset{\sim}{f}_\mu$ so is $(\mu\geqslant\lambda)$

(2) $\underset{\sim}{f}_{\underset{\cdot}{\circ}}(A_\lambda)=\underset{\sim}{f}_\lambda(A_\lambda)$.

We have

Theorem 2.3 Let $\underset{\sim}{f}:A\to B$ be a fuzzy mapping. then

(1) $\underset{\sim}{f}$ is injective iff φ and $\underset{\sim}{f}_{\underset{\cdot}{\circ}}:A_{\underset{\cdot}{\circ}}\to B_{\underset{\cdot}{\circ}}$ so are .

(2) $\underset{\sim}{f}$ is surjecfive iff φ so is and $\forall\ \lambda\in[0,h(A)]$, $\underset{\sim}{f}_{\underset{\cdot}{\circ}}:A_\lambda\to B_{\varphi(\lambda)}$ is surjective, i.e. $\underset{\sim}{f}_{\underset{\cdot}{\circ}}(A)=B$.

(3) $\underset{\sim}{f}$ is bijective iff φ and $\underset{\sim}{f}_{\circ}$ so are.

Corollary 2.4 Let $\underset{\sim}{f}=\langle f,\varphi\rangle:A\to B$ be a fuzzy mapping, $[0,h(A)]=[0,h(B)]=[0,1]$, φ be the identical mapping on $[0,1]$. Then,

(1) $\underset{\sim}{f}$ is injective (surjective, bijective) iff $\forall\ \lambda\in[0,1]$, $\underset{\sim}{f}_{\lambda}:A_{\lambda}\to B_{\lambda}$ so is.

(2) $\underset{\sim}{f}$ is injective(surjective, bjiective) iff $\forall\ \lambda\in[0,1)$, $\underset{\sim}{f}_{\lambda}:A_{\lambda}\to B_{\lambda}$ so is.

This is the theorem 5 in [1].

Theorem2.5 Let $\underset{\sim}{f}:A\to B$ be a fuzzy bijection, then $\underset{\sim}{f}$ images the shell—point in A to the shell—point in B. i.e. $\forall\ a_{A(a)}\in A, B(f(a))=\varphi(A(a))$.

Proof: Suppose $\underset{\sim}{f}$ is a fuzzy bijection. $\forall\ a_{A(a)}\in A$, $\underset{\sim}{f}(a_{A(a)})=f(a)_{\varphi(A(a))}\in B$. thus $\varphi(A(a))\leqslant B(f(a))$. Denoting $B(f(a))=\lambda_{\circ}$, then $f(a)\in B_{\lambda_{\circ}}$.

But $\underset{\sim}{f}$ is bijecfive implies φ is bijective, so exists $t\in Im(A)$, such that $\varphi(t)=\lambda_{\circ}$. Other hand, $f(a)_{\varphi(A(a))}=\underset{\sim}{f}(a_{A(a)})=\bigvee_{t\in Im(A)}\{f(a)_{\varphi(t)}\mid f_t(a)=f(a)\in B_{\varphi(t)}\}\geqslant f(a)_{\lambda_{\circ}}$, thus $\varphi(A(a))\geqslant\lambda_{\circ}$, So $\varphi(A(a))=\lambda_{\circ}=B(f(a))$, $f(a)_{\varphi(A(a))}=\underset{\sim}{f}(a_{A(a)})$ is a shell—point in B.

Definition 2.4. Let $\underset{\sim}{f}:A\to B$ be a fuzzy bijection, then a fuzzy bijection from B to A can be defined by $\underset{\sim}{f}$.

$\underset{\sim}{f}^{-1}:B\to A$, $\underset{\sim}{f}^{-1}(b_{\mu})=[f^{-1}(b)]_{\varphi^{-1}(\mu)}$ i.e. $\underset{\sim}{f}^{-1}=\langle f^{-1},\varphi^{-1}\rangle$.

We call $\underset{\sim}{f}^{-1}$ the fuzzy inverse mapping of $\underset{\sim}{f}$. It is clear, $(\underset{\sim}{f}^{-1})^{-1}=\underset{\sim}{f}$.

Definition 2.5 Let $\underset{\sim}{f}=\langle f,\varphi\rangle:A\to B$, $\underset{\sim}{g}=\langle g,\Psi\rangle:B\to C$ be fuzzy mappings, then a fuzzy mapping from A to C can be defined. $\underset{\sim}{g}\circ\underset{\sim}{f}:A\to C$, $\forall\ a_{\lambda}\in A$, $\underset{\sim}{g}\circ\underset{\sim}{f}(a_{\lambda})=\underset{\sim}{g}[\underset{\sim}{f}(a_{\lambda})]=[g(f(a))]_{\Psi(\varphi(\lambda))}$ and $\underset{\sim}{g}\circ\underset{\sim}{f}$ is called the composition mapping of $\underset{\sim}{f}$ and $\underset{\sim}{g}$.

Theorem 2.6. Let $\underset{\sim}{f}:A\to B$, $\underset{\sim}{g}:B\to C$ be fuzzy mappings and $\underset{\sim}{h}=\underset{\sim}{g}\circ\underset{\sim}{f}$,

then (1) $\underset{\sim}{f}$ and $\underset{\sim}{g}$ are both injective implies $\underset{\sim}{h}$ so is;

(2) $\underset{\sim}{f}$ and $\underset{\sim}{g}$ are both surjective implies $\underset{\sim}{h}$ so is;

(3) $\underset{\sim}{f}$ and $\underset{\sim}{g}$ are both bijective implies $\underset{\sim}{h}$ so is and $\underset{\sim}{h}$ is invertible. In addition, $\underset{\sim}{h}^{-1}=\underset{\sim}{f}^{-1}\circ\underset{\sim}{g}^{-1}$.

Theorem 2.7. Let $\underset{\sim}{f}:A\to B$, $\underset{\sim}{g}:B\to C$ be fuzzy mappings and $\underset{\sim}{h}=\underset{\sim}{g}\circ\underset{\sim}{f}$,

then (1) $\underset{\sim}{h}$ is injective implies $\underset{\sim}{f}$ so is.

(2) $\underset{\sim}{h}$ is surjective implies $\underset{\sim}{g}$ so is.

3. Application for fuzzy cardinal number

In[2], Li Hongxing etc have defined the fuzzy cardinal number:

Definition 3.1: [2] Let $A\in\mathscr{F}(X)$. $B\in\mathscr{F}(Y)$, we call A and B are equivalent if existing a fuzzy bijection between A and B, donoted $A\sim B$.

Obviously, the equivalence of L—fuzzy sets is an equivalent relation. Using this relation we can classify the all fuzzy sets in X, and the class contains A is called the cardinal number of A, denoted $|A|$. The fuzzy cardinal number is called F cardinal for short.

Theorem3.1. $A\sim B$ iff (1) $A_{\dot{\circ}}\sim B_{\dot{\circ}}$; (2) there exists a fuzzy mapping $\underset{\sim}{f}:A\to B$, such that $\forall\ a_{A(a)}\in A, B(f(a))=\varphi(A(a))$.

Theorem3.2. $A\sim B$ iff (1) $A_{\dot{\circ}}\sim B_{\dot{\circ}}$; (2) there exists a fuzzy mapping $\underset{\sim}{f}=\langle f,\varphi\rangle:A\to B$, such that $\forall\ a_{A(a)} \in A, B(f(a)) = \varphi(A(a))$.

Theorem3.3 $A \sim B$ iff existing a order-preserving mapping φ from Im(A) to Im(B), such that $\forall \lambda \in Im(A)$, $|A_\lambda| = |B_{\varphi(\lambda)}|$.

Proof: Necessity is clear.

(Sufficiency) $\forall \lambda, \mu \in Im(A)$, if $\mu \leqslant \lambda$, then $A_\mu \supseteq A_\lambda$, $B_{\varphi(\mu)} \supseteq B_{\varphi(\lambda)}$. By $|A_\lambda| = |B_{\varphi(\lambda)}|$ and $|A_\mu| = |B_{\varphi(\mu)}|$, the bijection f_λ from A_λ to $B_{\varphi(\lambda)}$ and the bijection σ_μ from A_μ to $B_{\varphi(\mu)}$ can be defined respectively. Since $\sigma_\mu|_{A_\lambda}$ is bijective also, thus we can define an bijection $\varphi_1 = f_\lambda \circ \sigma_\mu^{-1}: \sigma_\mu(A_\lambda) \to B_{\varphi(\lambda)}$ and a bijection $\varphi_2: B_{\varphi(\mu)} \backslash \sigma_\mu(A_\lambda) \to B_{\varphi(\mu)} \backslash B_{\varphi(\lambda)}$.

Let $\theta: B_{\varphi(\mu)} \to B_{\varphi(\mu)}, \theta(b) = \begin{cases} \varphi_1(b) & b \in \sigma_\mu(A_\lambda) \\ \varphi_2(b) & b \in B_{\varphi(\mu)} \backslash \sigma_\mu(A)_\lambda \end{cases}$,

then θ is bijective.

Let $f_\mu = \theta \circ \sigma_\mu$, then $\forall a \in A_\mu$,

$$f_\mu(a) = \theta \circ \sigma_\mu(a) = \begin{cases} f_\lambda(a) & a \in A_\lambda \\ \varphi_2(\sigma_\mu(a)) & a \in A_\mu \backslash A_\lambda. \end{cases}$$

f_μ is bijection from A_μ to $B_{\varphi(\mu)}$ and $f_\mu|_{A_\lambda} = f_\lambda$.

According to this way, we can construct the bijections $f_{h(A)}, \cdots\cdots f_{\overset{\circ}{.}}$. Let $f = f_{\overset{\circ}{.}}$, $\underset{\sim}{f} = \langle f_{\overset{\circ}{.}}, \varphi \rangle$, then $\underset{\sim}{f}$ is a bijection from A to B. Thus we have: $A \sim B$.

Theorem3.3. Let A and B be fuzzy sets have finite supports, then

(1) $A \sim B$ iff existing a bijection φ from $[0, h(A)]$ to $[0, h(B)]$, such that $\forall \lambda \in [0, h(A)]$, $\sum_{x \in A_\lambda} A(x) = \sum_{y \in B_{\varphi(\lambda)}} B(y)$;

(2) $A \sim B$ iff existing a bijection φ from $[0, h(A)]$ to $[0, h(B)]$, such that $\forall a \in A_{\overset{\circ}{.}}$, $\sum_{A(x) = A(a)} A(x) = \sum_{B(y) = \varphi(A(a))} B(y)$.

Obviously, the theorem4.1 and its corollary in[2] are the special cases of the above conclutions.

References

[1]. Li Hongxing etc ,How to define mappings of fuzzy sets,Journal of BeiJing Normal University,Vol. 29,No. 1(1993),1—9.

[2]. Li Hongxing etc, Fuzzy mappings and fuzzy cardinal numbers, Applied Mathematics — A Journal of Chinese Univer sities, Vol. 9,Ser. A No. 2 (1994), 177—186.

[3]. Li Hongxing etc,Fuzzy Cardinality and Continuum Hypothesis,Journal of Mathematical Research and Exposition,Vol. 15,No,1(1995),129—134.

[4]. Zhang Chengyi,The pointwise characterization of fuzzy mappings,chinese Quarterly Journal of Mathematics,(to appear).

[5]. Zhang Chengyi, The Expressions and Comparisions of Fuzzy cardinal numbers, Applied Mathematics—A Journal of chinese Universities,(to appear)

[6]. Luo Chengzhong . Introdiction of fuzzy sets ,Publishing company of Beijing Normal University , 1989.