The weak induced fuzzy ring and fuzzy σ-ring

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Abstract: The weak induced fuzzy ring and the weak induced fuzzy σ-ring are defined and their fundamental properties are studied.
Keywords: Weak induced fuzzy ring; weak induced fuzzy σ-ring; the monotone class of fuzzy set.

1. Introduction and Preliminaries

Since 1965 the concept of fuzzy set was induced by L. A. Zadeh, many fundamental mathematical concepts have been generalized such as fuzzy topological space[3] and fuzzy group[4] and they are studied deeply. As well known, the measure theory is estiablished under the concept ring and σ-ring(algebra and σ-algebra), but the theory of ring and σ-ring of fuzzy sets are not studied deeply which is necessary when the measure of fuzzy sets is studied. The purpose of this paper is to study the structure of ring and σ-ring of fuzzy sets. Following we give some basic well known definitions.

Definition 1.1 [1] Let R be a nonempty class which elements are sets. If R can satisfy the following conditions:
if $E \in R$, $F \in R$, then $E \cup F \in R$, $E - F = E \cap F^c \in R$,
then R is called a ring.

Definition 1.2 [1] Let R be a nonempty class which elements are sets. If R can satisfy the following conditions:

if $E_i \in R$, then $\bigcup_{i=1}^{\infty} E_i \in R$; if $E, F \in R$, then $E - F \in R$;

then R is called a σ-ring.

Definition 1.3 [1] Let M be a nonempty class which elements are sets. If for any monotonic sequence $\{E_n\}$, $E_n \in M$, n=1,2,..., we have $\lim_{n\to\infty} E_n \in M$, then M is called a monotone class.

Definition 1.4 [2] A fuzzy set A is defined as a function A: $X \to [0,1]$, X is a classical set, and $A_r = \{x \in X: A(x) \geq r\}$ is called the r-level cut and $l_r(A) = \{x \in X: A(X) > r\}$ is called the strong level cut of A. And we have $A=\cup_{r\in[0,1]} r\cdot A_r=\cup_{r\in[0,1]} r\cdot l_r(A)$.

2.The weak induced fuzzy ring

Definition 2.1 Let R be a nonempty class which elements are fuzzy sets of X. If it can satisfy the following conditions:
if $E\in R$, $F\in R$, then $E\cup F\in R$, $E-F\triangleq E\cap F^C\in R$,
then R is called a fuzzy ring, here $F^C(x)=1-F(x)$, $x\in X$.

If A is a fuzzy set, then $A\cap A^C=\phi$ and $A\cup A^C=X$ is not always true, and many useful properties of the classical ring can not be kept such as $\phi\in R$ is not always true. So we think about a special kind of fuzzy ring which has better properties.

Definition 2.2 Let M be a fuzzy sets class. If for any $A\in M$, $r\in[0,1]$, $r\cdot X_{l_\alpha(A)}\in M$, $\alpha\in[0,1]$, then M is called having the weak induced property. Here $r\cdot X_{l_\alpha(A)}(x)=r$, $x\in l_\alpha(A)$; $r\cdot X_{l_\alpha(A)}(x)=0$, $x\notin l_\alpha(A)$.

Proposition 2.3 Let R be a weak induced fuzzy ring, then for any $\alpha\in[0,1)$, $l_\alpha(R)=\{l_\alpha(A): A\in R\}$ is a classical ring, and for any $\alpha\in[0,1)$, $l_\alpha(R)$ is invariant which means for any $\alpha_1,\alpha_2\in[0,1)$, $\alpha_1\neq\alpha_2$, we have $l_{\alpha_1}(R)=l_{\alpha_2}(R)$.

Proof Let $\alpha\in[0,1)$, $l_\alpha(A), l_\alpha(B)\in l_\alpha(R)$, then there exists $\varepsilon>0$ such that $\varepsilon+\alpha<1$, and $(\alpha+\varepsilon)\cdot X_{l_\alpha(A)}$, $(\alpha+\varepsilon)\cdot X_{l_\alpha(B)}\in R$, hence we have $(\alpha+\varepsilon)\cdot X_{l_\alpha(A)}\cup(\alpha+\varepsilon)\cdot X_{l_\alpha(B)}=(\alpha+\varepsilon)\cdot X_{l_\alpha(A)\cup l_\alpha(B)}\in R$, but $l_\alpha((\alpha+\varepsilon)\cdot X_{l_\alpha(A)\cup l_\alpha(B)})=l_\alpha(A)\cup l_\alpha(B)$, hence $l_\alpha(A)\cup l_\alpha(B)\in l_\alpha(R)$.
We also have

$$((\alpha+\varepsilon)\cdot X_{l_\alpha(A)}-(\alpha+\varepsilon)\cdot X_{l_\alpha(B)})(x)=\begin{cases}0, x\notin l_\alpha(A)\cup l_\alpha(B);\\ \alpha+\varepsilon, x\in l_\alpha(A)-l_\alpha(B);\\ 0, x\in l_\alpha(B)-l_\alpha(A);\\ (\alpha+\varepsilon)\wedge(1-(\alpha+\varepsilon)), x\in l_\alpha(A)\cap l_\alpha(B);\end{cases}$$

i) If $\alpha\geqslant 0.5$, then $\alpha+\varepsilon>\alpha>1-(\alpha+\varepsilon)$, so
$l_\alpha((\alpha+\varepsilon)\cdot X_{l_\alpha(A)}-(\alpha+\varepsilon)\cdot X_{l_\alpha(B)})=l_\alpha(A)-l_\alpha(B)$.
Since $(\alpha+\varepsilon)\cdot X_{l_\alpha(A)}-(\alpha+\varepsilon)\cdot X_{l_\alpha(B)}\in R$, hence $l_\alpha(A)-l_\alpha(B)\in l_\alpha(R)$.
ii) If $\alpha<0.5$, then $2\alpha<1$, take $\varepsilon>1-2\alpha$, then $\alpha>1-(a+\varepsilon)$, so like i) we have $l_\alpha(A)-l_\alpha(B)\in l_\alpha(R)$. That is to say $l_\alpha(R)$ is a ring.
For any $\alpha\in[0,1)$, $l_\alpha(R)$ is invariant is clear.

Definition 2.4 Let R be a fuzzy set class. If R satisfy the following conditions

(a) if $E \in R, F \in R$, then $E \bigcup F \in R$;

(b) if $E \in R$, then $E^C \in R$;

then R is called a fuzzy algebra.

Proposition 2.5 If $X \in R$ and R is a fuzzy ring, then R is a fuzzy algebra.
If R is a fuzzy algebra, then X is not always in R.

Proof Since $A^c = X - A$, then R is a fuzzy algebra.
Let $R=\{A\}$, $A(x)=0.5$, $x \in X$, then R is a fuzzy algebra, but $X \notin R, \phi \notin R$.

If R is a weak induced fuzzy ring, it is clear if R is a fuzzy algebra, then $X \in R$.

3. The fuzzy ring and fuzzy σ-ring

Proposition 3.1 Let R_t be fuzzy rings (weak induced fuzzy rings), $t \in T$, then $\bigcap_{t\in T} R_t$ is a fuzzy ring (weak induced fuzzy ring) if $\bigcap_{t\in T} R_t \neq \phi$.

Proof Let $R=\bigcap_{t\in T} R_t$, for any $A, B \in R$, we have $A, B \in R_t$, hence $A \bigcup B \in R_t, A - B \in R_t$, $t \in T$, hence $A \bigcup B \in R, A - B \in R$.

Proposition 3.2 Let E be an arbitrary fuzzy set class, then there is a only one fuzzy ring R_0 such that $R_0 \supseteq E$, and for any fuzzy ring R, $R \supseteq E$, we have $R_0 \subseteq R$.

Proof Since the fuzzy set class which include all the fuzzy sets of X is a fuzzy ring, so there is at least one fuzzy ring which include E. And by Proposition 3.1 we know the intersection of all the fuzzy rings which include E is also a fuzzy ring, hence the intersection of all the fuzzy rings which include E is the R_0 which we want to get. We denote it as $R(E)$.

Proposition 3.3 Let E be a fuzzy sets class, $A \in R(E)$. Then there are $A_i \in E\,(1 \le i \le n)$ such that $A \subseteq \bigcup_{i=1}^n A_i$.

Proof Let $R=\{\mathrm{A}:\mathrm{A} \subseteq \bigcup_{i=1}^n A_i, A_i \in E, i \le n\}$, it is clear that R is a fuzzy ring, and $E \subseteq R$, hence $R(E) \subseteq R$.

Proposition 3.4 Let E be a countable class of fuzzy sets, then $R(E)$ is also countable.
The proof is as same as the case of classical set.

Definition 3.5 Let S be a nonempty class which elements are fuzzy set. If S satisfies the following conditions:

(a) if $E \in S$, $F \in S$, then $E - F \in S$;

(b) if $E_i \in S$, i=1,2,…, then $\bigcup_{i=1}^{\infty} E_i \in S$;

we call S a fuzzy σ-ring.

Proposition 3.6 The weak induced fuzzy σ-ring is closed under the operation of countable intersection.

Proof Let S be a weak induced fuzzy σ-ring, $E_i \in S$, $i \in N$. For any $\alpha \in [0,1]$, first we prove for any $r \in [0,1]$, $r \cdot X_{(E_i)_\alpha} \in S$.

Let $r_n \to \alpha, r_n < \alpha$, then $(E_i)_\alpha = \bigcap_{n=1}^{\infty} l_{r_n}(E_i)$.

Let $E = \bigcup_{n=1}^{\infty} X_{l_{r_n}(E_i)}$, then $E \in S$, and $X_{(E_i)_\alpha} = E - \bigcup_{n=1}^{\infty}(E - X_{l_{r_n}(E_i)}) \in S$.

Hence for any $r \in [0,1]$, $r \cdot X_{(E_i)_\alpha} \in S$.

Similarly we can prove for any $r \in [0,1]$, $r \cdot X_{\bigcap_{i=1}^{\infty}(E_i)_\alpha} \in S$.

Supposing α take over the set of all the rational number in $[0,1]$, then we have

$$\bigcap_{i=1}^{\infty} E_i = \bigcup_{\alpha \in [0,1]} \alpha \cdot X_{(\bigcap_{i=1}^{\infty} E_i)_\alpha} = \bigcup_{\alpha \in [0,1]} \alpha \cdot X_{\bigcap_{i=1}^{\infty}(E_i)_\alpha} \in S.$$

Proposition 3.7 Let S_0 be a σ-classical ring, $S^* = \{A : l_r(A) \in S_0,\ r \in [0,1]\}$ then S^* is a weak induced fuzzy σ-ring.

Proof If $E_i \in S^*$, $i=1,2,\ldots$, then $l_\alpha(\bigcup_{i=1}^{\infty} E_i) = \bigcup_{i=1}^{\infty} l_\alpha(E_i) \in S_0$, hence $\bigcup_{i=1}^{\infty} E_i \in S^*$.

If $E, F \in S^*$, for any $\alpha \in [0,1]$, then

$$\begin{aligned} l_\alpha(E - F) &= l_\alpha(E) \cap l_\alpha(F^c) \\ &= l_\alpha(E) \cap (F_{1-\alpha})^c \\ &= l_\alpha(E) \cap (\bigcap_{\beta < 1-\alpha} l_\beta(F))^c \\ &= l_\alpha(E) \cap (\bigcup_{\beta < 1-\alpha} ((l_\beta(F))^c) \\ &= \bigcup_{\beta < 1-\alpha} (l_\alpha(E) - l_\beta(F)). \end{aligned}$$

Let $r_n \to 1 - \alpha$, $r_n < 1 - \alpha$, then

$$l_\alpha(E - F) = \bigcup_{n=1}^{\infty} (l_\alpha(E) - l_{r_n}(F)) \in S_0$$

hence $E - F \in S^*$, S^* is a fuzzy σ-ring.

S^* is weak induced is clear.

4. The monotone class of fuzzy sets

Definition 4.1 Let M be a fuzzy set class. If for any monotonic sequence $\{E_n\} \subseteq M$, $\lim_{n \to \infty} E_n \in M$, then M is call a fuzzy monotone class.

Proposition 4.2 A weak induced fuzzy σ-ring is a fuzzy monotone class; and a monotone fuzzy ring is a fuzzy σ-ring.

Proof Since the weak induced fuzzy σ-ring is closed under the operation of countable union and countable intersection, we know the weak induced fuzzy σ-ring is a fuzzy monotone class.

Let M be a monotone fuzzy ring, $E_i \in M$, i=1,2,..., so $\bigcup_{i=1}^{n} E_i \in M$, i=1,2,..., Since $\{\bigcup_{i=1}^{n} E_i\}$ is a monotonic sequence, and $\bigcup_{n=1}^{\infty}(\bigcup_{i=1}^{n} E_i)\} = \bigcup_{i=1}^{\infty} E_i$, hence we get $\bigcup_{i=1}^{\infty} E_i \in M$.

Theorem 4.3 Let R be a weak induced fuzzy ring. Let $R^*=\{A:l_r(A)\in S(l_r(R))\}$, $M^*(R)=\{A:l_r(A)\in M(l_r(R))\}$, then $M^*(R)$ is a weak induced fuzzy monotone class and $R^*=M^*(R)$. Here $l_r(R)=\{l_r(A):A\in R\}$, $S(l_r(R))$ is the classical σ-ring generated by $l_r(R)$, $M(l_r(R))$ is the classical monotone class generated by $l_r(R)$.

Proof R^* is a weak induced fuzzy σ-ring.

For any $\{A_n\}\subseteq M^*(R), A_1\subseteq A_2\subseteq\cdots\subseteq A_n\subseteq\cdots$, we have

$\lim_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} A_n$. For any $r\in[0,1]$, $l_r(\bigcup_{n=1}^{\infty} A_n) = \bigcup_{n=1}^{\infty} l_r(A_n)\in M(l_r(R))$, hence

$\bigcup_{n=1}^{\infty} A_n \in M^*(R)$.

Let $\{A_n\}\subseteq M^*(R)$, $A_1\supseteq A_2\supseteq\cdots\supseteq A_n\supseteq\cdots$, we have

$\lim_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} A_n$. And for any $r\in[0,1]$, $(\bigcap_{n=1}^{\infty} A_n)_r = \bigcap_{n=1}^{\infty}(A_n)_r$. But

$(A_n)_r=\bigcap_{m=1}^{\infty} l_{r_m}(A_n)$, $r_m \to r, r_m < r$, hence $(A_n)_r\in M(l_r(R))$, and $(\bigcap_{n=1}^{\infty} A_n)_r \in M(l_r(R))$.

We also have $l_r(\bigcap_{n=1}^{\infty} A_n) = \bigcup_{i=1}^{\infty}(\bigcap_{n=1}^{\infty} A_n)_{r_i}, r_i \to r, r_i > r$, hence $\bigcap_{n=1}^{\infty} A_n \in M^*(R)$.

So $M^*(R)$ is a weak induced fuzzy monotone class.

Since $S(l_r(R))=M(l_r(R))$, then $R^*=M^*(R)$.

Remark 4.4. In 1983 Prof. Wang Ge Ping has studied the fuzzy σ-algebra which could be generated by an usual σ-algebra in [5], some of his opinions is similarly to ours.

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