A Note on Convex Fuzzy Sets

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Abstract

This note is to give the additional properties which are required to characterize convex sets.

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1. Introduction

Some properties of convex fuzzy sets were studied by Lowen [1], Liu [2] and yang [3,4]. In this note some additional properties which are required to characterize convex fuzzy sets are discussed. These additional properties are expressed in terms of the following line segment characterization of convex fuzzy sets and the monotonicity.

2. Preliminaries

Similar to [3] and [4], throughout this paper E will denote the n-dimensional Euclidean space R^n, I denotes the interval [0,1], I^0 denotes (0,1) and $[x, y]$ denotes set $\{ax+(1-a)y \mid a\in I\}$. Fuzzy sets and values in I will be denoted by lower case Greek letters and we shall make no difference between notations for a fuzzy set with a constant value and that value itself.

Definition 1. The fuzzy set λ on E is said to be a convex fuzzy set if and only if for all $x,y\in E$ and $a\in I$,

$$\lambda(ax+(1-a)y) \geq \lambda(x)\wedge\lambda(y)$$

It is easy to see that λ is a convex fuzzy set if and only if for all $a\in I$, $\lambda^{-1}[a,1]$ is convex.

3. Main results

This section gives the main results of this note. That is, we obtain some additional properties which are required to characterize convex fuzzy sets.

Theorem 1. The fuzzy set λ on E is a convex fuzzy set if and only if for all line l in E, λ is a convex fuzzy set on $l\cap E$.

The proof is straightforward by Definition 1.

Theorem 2. The fuzzy set λ on E is a convex fuzzy set if and only if for all $x,, y\in E$, there exist intervals A ($0\in A$) and B ($0\in B$), such that $f(a) = \lambda(ax+(1-a)y)$ is a increasing function on A, is a descent function on B, and $A\cup B\supseteq I$.

Proof. Let λ be a convex fuzzy set on E.

(i) For $\forall\ x, y\in E$, let

$$A = \{a\in I \mid f(b)\leq f(a),\ \forall\ 0\leq b\leq a\} \qquad (1)$$

and

$$\sup A = a_2 \qquad (2)$$

Without loss of generality, suppose that $a_2 > 0$, now we prove that $[0, a_2)\subseteq A$. In fact, if that is not true, then there exists a $b\in[0, a_2)$ such that $b\notin A$, that is ,there exist $0\leq b_2<b$ such that $f(b_2) > f(b)$. Form (2), there exists a $b_1\in A$ such that $b_1>b$. Consequently, we have $f(b_1) \geq f(b_2) > f(b)$, that is,

$$f(b) < f(b_1) \wedge f(b_2) \qquad (3)$$

Now, let $c = \dfrac{b-b_1}{b_1-b_2}$, $x_1 = x+b_2(y-x)$ and $y_1 = x+b_1(y-x)$, we have $c\in I^0$ and

$$cy_1+(1-c)x_1 = x_1+c(y_1-x_1) = x+b(y-x)$$

Also, λ is a convex fuzzy set, it follows that

$$f(b) = \lambda(x+b(y-x)) = \lambda(cy_1+(1-c)x_1) \geq \lambda(x_1)\wedge\lambda(y_2) = f(b_1) \wedge f(b_2)$$

contradicting (3). Thus, $[0, a_2)\subseteq A$.

(ii) For $\forall\ x,y\in E$, let

$$A = \{\ b\in I \mid\ \lambda(y+b(x-y)) \geq \lambda(y+b_1(x-y)),\ \forall\ 0\leq b_1\leq b\}$$

and

$$\sup B=b_2$$

Similar to (i), we can show that $[0, b_2)\subseteq B$. Commuting x to y, and y to x, we have $[a_1,1)\subseteq B$, where

$a_1 = 1-b_1$. So B is a interval and $f(a) = \lambda(x+b(y-x))$ is a descent function on B.

(iii) Now we prove that $A\cup B\supseteq I$. If that is not true, then there exists a $a\in I$ such that $a\notin A\cup B$. It follows that there exists a $a'\in I$ and $a'<a$ such that $f(a') > f(a)$ by $a\notin A$, there exists a $a''\in I$ and such that $f(a'') > f(a)$ by $a\notin B$. So we have $f(a)<f(a')\wedge f(a'')$

Similar to (i), this leads to a contradiction.

Theorem 3. Let λ be a convex fuzzy set on E. If there exists a $a_0\in I^0$ such that

$$\lambda(a_0x+(1-a_0)y) \geq \lambda(x)\wedge\lambda(y),\ \forall x,\ y\in E$$

Then set

$$M = \{a\in I^0 \mid \lambda(ax+(1-a)y) \geq \lambda(x)\wedge\lambda(y),\ \forall x,\ y\in E\}$$

is infinite and dense everywhere in I^0.

Proof. $\forall x,\ y\in E$, since $a_0\in M$ and

$$\lambda((1-a_0)x+a_0y) = \lambda(a_0y+(1-a_0)x) \geq \lambda(x)\wedge\lambda(y)$$

we have $1\text{-}a_0 \in M$. Now, $a_0x+(1\text{-}a_0)y\in E$, and

$$a_0[a_0x+(1\text{-}a_0)y]+(1\text{-}a_0)y = a_0^2x+(1\text{-}a_0^2)y$$

Therefore, $\lambda[a_0^2x+(1\text{-}a_0^2)y]\geq\lambda[a_0x+(1\text{-}a_0)y]\wedge\lambda(y)\geq \lambda(x)\wedge\lambda(y)$

that is, $a_0^2 \in M$. It follows that a_0^k, $(1\text{-}a_0^k)\in M$, $\forall k\in\mathcal{N}$ by mathematical induction. Thus, M is infinite.

Now, we prove that M is dense everywhere in I^0. If that is not true, then there exist $\bar{a}\in I^0$, $U(\bar{a}) = \{a\in M \mid a\geq\bar{a}\}$, $V(\bar{a}) = \{a\in M \mid a\leq\bar{a}\}$ such that $U(\bar{a})\neq\varnothing$, $V(\bar{a})\neq\varnothing$ and

$$\inf U(\bar{a})=\bar{a}_2>\bar{a}_1=\sup V(\bar{a})$$

that is, $\quad (\bar{a}_1,\bar{a}_2)\cap M=\varnothing \qquad (4)$

Choose $a_1\in V(\bar{a})$ and $a_2\in U(\bar{a})$ such that $\bar{a}_1<a_3=a_1+a_0(a_2\text{-}a_1)<\bar{a}_2$. Since $a_1,a_2\in M$, for $\forall x$, $y\in E$, we have

$$\begin{aligned}&\lambda[a_3x+(1\text{-}a_3)y]\\ &=\lambda[a_0(a_2x+(1\text{-}a_2)y)+(1\text{-}a_0)(a_2x+(1\text{-}a_2)y)]\\ &\geq\lambda[a_2x+(1\text{-}a_2)y]\wedge\lambda[a_1x+(1\text{-}a_1)y]\\ &\geq [\lambda(x)\wedge\lambda(y)]\wedge [\lambda(x)\wedge\lambda(y)]\\ &=\lambda(x)\wedge\lambda(y)\end{aligned}$$

Consequently, we have $a_3\in M$ which contradict (4).

Corollary Let λ be a convex fuzzy set on E. If there exists a $a_0\in I^0$ such that

$$\lambda(a_0x+(1\text{-}a_0)y)\geq \lambda(x)\wedge\lambda(y),\ \forall x,\ y\in E$$

Then for all $x,y\in E$, there exists a set X, which is dense everywhere on $[x,y]$, such that

$$\lambda(z)\geq \lambda(x)\wedge\lambda(y),\ \forall\ z\in X$$

References

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