The algebraic structure of the sets of all ideals with the same tip

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Abstract: In this paper, we investigate the algebraic structure of the set $F_i(R)$ of all fuzzy ideals and the sets $F_i^t(R)$ of all fuzzy ideals with the same tip "t" for a ring R. Then we discuss the relations of these algebraic systems by the theory of homomorphism and isomorphism.
Keywords: equivalence relation, partition, quotient set, semigroup, lattice.

1. Preliminaries

In this section, we give some definitions and results which will be used in the sequel.

Throughout this paper, R always represents a ring.

Let X be a set. A fuzzy set μ in X is a map $\mu: X \to [0, 1]$. F(X) denotes the set of all fuzzy set of X.

Definition 1.1 Let $\mu \in F_i(R)$. If for any x, y $\in$ R,

(1) $\mu(x-y) \geqslant \mu(x) \wedge \mu(y)$,

(2) $\mu(xy) \geqslant \mu(x) \wedge \mu(y)$,

then we call μ a fuzzy subring of R.

Definition 1.2 Let μ be a fuzzy subring of R. The tip of μ means $\mu(0)$, the value at the addition unit 0 of the given ring R.

Definition 1.3 Let μ be a fuzzy subring of R. If for any x, y $\in$ R,

$\mu(xy) \geqslant \mu(x) \vee \mu(y)$,

then we call μ a fuzzy ideal of R.

If μ is a fuzzy subring of R,then $\mu(0) \geqslant \mu(x)$, $x \in R$; If R has product unit e, then $\mu(e) \geqslant \mu(x)$ for $x \in R$ and $x \neq 0$.

We denotes the set of all fuzzy ideals of R by the symbol $F_i(R)$. We define the binary relation $\sim$ of $F_i(R)$ as follows:

$$\mu \sim \eta \Leftrightarrow \mu(0) = \eta(0).$$

Obviously, $\sim$ be a equivalent relation.

Let $F_i^t(R) = \{\mu | \mu \in F_i(R) \ and \ \mu(0) = t\}$, then the partition of $F_i(R)$ determined by $\sim$ is: $F_i(R)/\sim = \{F_i^t(R) | t \in [0,1]\}$.

2. The operations on $F_i(R)$ and $F_i^t(R)$ and their algebraic structure

We will study the algebraic structure of $F_i(R)$ and $F_i^t(R)$ from the respect of lattice, sum $\mu + \eta$, product $\mu \circ \eta$.

Theorem 2.1 $F_i(R)$ forms a complete lattice under the ordering of fuzzy set inclusion $\leqslant$.

Definition 2.1 Let μ and η be a fuzzy sets of R. We define the sum μ + η as follows:

$$(\mu+\eta)(x)=\vee\{\mu(y)\wedge\eta(z) \mid y+z=x\},\ x\in R.$$

Theorem 2.2 $F_i(R)$ forms a monoid under the sum + with the unit $1_{\{0\}}$, denoted by $(F_i(R), +)$.

Proof. Omitted. □

Author investigated the lattice structure of $F_i'(R)$ in [2] and obtained the following results:

Theorem 2.3 $F_i'(R)$ forms a complete and modular lattice under the ordering of fuzzy set inclusion ⩽, denoted by $(F_i'(R), \vee, \wedge)$, and have

$$\mu\vee\eta=\mu+\eta.$$

Remark In $F_i(R)$, $\mu\vee\eta=\mu+\eta$ is not right.

Theorem 2.4 $F_i'(R)$ forms a monoid to the sum + with the unit $t_{\{0\}}$, denoted by $(F_i'(R), +)$.

Proof. Omitted. □

Definition 2.2 Let μ and η be fuzzy subsets of R, we define the product $\mu\circ\eta$ as follows:

$$(\mu\circ\eta)(x)=\vee\left\{\mathop{\wedge}_{i=1}^{n}(\mu(y_i)\wedge\eta(z_i))\,\middle|\, n\in N, \sum_{i=1}^{n}y_iz_i=x\right\}.$$

It is obviously that above definition 2.2 is the generalization of the product in the classicall ring theory.

Theorem 2.5 $F_i'(R)$ forms a semigroup to the product ∘, denoted by $(F_i'(R), \circ)$. If R has the unit e, then $(F_i'(R), \circ)$ forms a monoid with the unit $t_{\{e\}}$.

Proof. For any $\mu,\ \eta\in F_i'(R)$,

$$(\mu\circ\eta)(0)=\vee\left\{\mathop{\wedge}_{i=1}^{n}(\mu(y_i)\wedge\eta(z_i))\,\middle|\, n\in N, \sum_{i=1}^{n}y_iz_i=0\right\}$$
$$\geqslant\mu(0)\wedge\eta(0)$$
$$=t,$$

by $\mu(y_i)\le t$, $\eta(z_i)\le t$, so $(\mu\circ\eta)(0)\le t$, hence $(\mu\circ\eta)(0)=t$.

For any x, y ∈ R,

$$(\mu\circ\eta)(x)\wedge(\mu\circ\eta)(y)$$
$$=\left(\vee\left\{\mathop{\wedge}_{i=1}^{n}(\mu(y_i)\wedge\eta(z_i))\,\middle|\, n\in N, \sum_{i=1}^{n}y_iz_i=x\right\}\right)\wedge\left(\vee\left\{\mathop{\wedge}_{j=n+1}^{n+m}(\mu(y_j)\wedge\eta(z_j))\,\middle|\, m\in N, \sum_{j=n+1}^{n+m}y_jz_j=y\right\}\right)$$
$$=\vee\left\{\mathop{\wedge}_{i=1}^{n}(\mu(y_i)\wedge\eta(z_i))\wedge\left(\mathop{\wedge}_{j=n+1}^{m+n}(\mu(y_j)\wedge\eta(z_j))\right)\,\middle|\, m,n\in N, \sum_{i=1}^{n}y_iz_i=x, \sum_{j=n+1}^{m+n}y_jz_j=y\right\}$$
$$\le\vee\left\{\mathop{\wedge}_{i=1}^{m+n}(\mu(y_i)\wedge\eta(z_i))\,\middle|\, m,n\in N, \sum_{i=1}^{m+n}y_iz_i=x+y\right\}$$
$$=(\mu\circ\eta)(x+y),$$

Obviously, $(\mu\circ\eta)(-x)=(\mu\circ\eta)(x)$,

$$(\mu\circ\eta)(xy)=\vee\left\{\overset{n}{\underset{i=1}{\wedge}}(\mu(y_i)\wedge\eta(z_i))\middle| n\in N,\sum_{i=1}^{n}y_iz_i=xy\right\}$$
$$\geqslant\vee\left\{\overset{n}{\underset{i=1}{\wedge}}(\mu(xy_i)\wedge\eta(z_i))\middle| n\in N,\sum_{i=1}^{n}(xy_i)z_i=xy\right\}$$
$$\geqslant\vee\left\{\overset{n}{\underset{i=1}{\wedge}}(\mu(y_i)\wedge\eta(z_i))\middle| n\in N,\sum_{i=1}^{n}y_iz_i=y\right\}$$
$$=(\mu\circ\eta)(y),$$

Similarly, $(\mu\circ\eta)(xy)\geqslant(\mu\circ\eta)(x)$,so

$$(\mu\circ\eta)(xy)\geqslant(\mu\circ\eta)(x)\vee(\mu\circ\eta)(y),$$

hence $\mu\circ\eta\in F_i^t(R)$, associative is right[3], these implies that $F_i^t(R)$ forms a semigroup to $\circ$.

If R has unit e, it is easy to know $t_{\{e\}}$ is the unit of ($F_i^t(R)$,$\circ$) . □

From the proof above we can also obtain the following result:

Definition 2.6 $F_i(R)$ forms a semigroup to $\circ$, denoted by ($F_i(R)$,$\circ$) . If R has unit e, ($F_i(R)$,$\circ$) forms a monoid with unit $1_{\{e\}}$.

3. Homomorphism and isormorphism

Let μ ∈ F(R), a ∈(0, ∞). If a(μ (x))∈[0, 1],for all x ∈ R, then we call a is productable with μ , and define the product a μ as follows:

(a μ)(x)=a(μ (x)), x ∈ R.

Lemma 3.1 Let a ∈(0, ∞), $b_i\in[0,1]$, i ∈ I, I is an index set, and $ab_i\in[0,1]$, then

$$a(\underset{i\in I}{\vee}b_i)=\underset{i\in I}{\vee}(ab_i)\ .$$

Theorem 3.2 Let s, t ∈(0, 1], then

(1) $(F_i^s(R),\ \vee,\ \wedge)\cong(F_i^t(R),\ \vee,\ \wedge)$,

(2) $(F_i^s(R),+)\cong(F_i^t(R),+)$,

(3) $(F_i^s(R),\ \circ)\cong(F_i^t(R),\ \circ)$

Proof. We define the map f from $F_i^s(R)$ to $F_i^t(R)$ as follows:

$$\text{f: } F_i^s(R)\to F_i^t(R)\ ,\quad \mu\mapsto\frac{t}{s}\mu.$$

For μ ∈ $F_i^s(R)$, obviously, $\frac{t}{s}$ is productable with μ , and $\frac{t}{s}\mu\in F_i^t(R)$. It is not difficult to verify that f is a bijection from $F_i^s(R)$ to $F_i^t(R)$.

(1) ∀ μ , η ∈ $F_i^s(R)$,

$$f(\mu\vee\eta)=\frac{t}{s}(\mu\vee\eta)=(\frac{t}{s}\mu)\vee(\frac{t}{s}\eta)=f(\mu)\vee f(\eta),$$

$$f(\mu \wedge \eta)=\frac{t}{s}(\mu \wedge \eta)=(\frac{t}{s}\mu)\wedge(\frac{t}{s}\eta)=f(\mu)\wedge f(\eta),$$

hence, $(F_i^s(R), \vee, \wedge)\cong(F_i^t(R), \vee, \wedge)$,

(2) $\forall \mu, \eta \in F_i^s(R), x \in R$,

$$f(\mu+\eta)(x)=(\frac{t}{s}(\mu+\eta))(x)=\frac{t}{s}((\mu+\eta)(x))$$
$$=\frac{t}{s}(\vee\{\mu(x_1)\wedge\eta(x_2)|x_1+x_2=x\})$$
$$=\vee\{(\frac{t}{s}\mu(x_1))\wedge(\frac{t}{s}\eta(x_2))|x_1+x_2=x\})$$
$$=\vee\{(\frac{t}{s}\mu)(x_1)\wedge(\frac{t}{s}\eta)(x_2)|x_1+x_2=x\})$$
$$=(\frac{t}{s}\mu+\frac{t}{s}\eta)(x)$$
$$=(f(\mu)+f(\eta))(x).$$

so, $(F_i^s(R), +)\cong(F_i^t(R), +)$. Obviously, $f(s_{\{0\}})=f(t_{\{0\}})$.

(3) $\forall \mu, \eta \in F_i^s(R), x \in R$,

$$f(\mu \circ \eta)(x)=(\frac{t}{s}(\mu \circ \eta))(x)=\frac{t}{s}((\mu \circ \eta)(x))$$
$$=\frac{t}{s}(\vee\{\bigwedge_{i=1}^{n}(\mu(y_i)\wedge\eta(z_i))\Big|n\in N, \sum_{i=1}^{n}y_iz_i=x\})$$
$$=\vee\{\frac{t}{s}(\bigwedge_{i=1}^{n}(\mu(y_i)\wedge\eta(z_i)))\Big|n\in N, \sum_{i=1}^{n}y_iz_i=x\}$$
$$=\vee\{\bigwedge_{i=1}^{n}(\frac{t}{s}\mu(y_i))\wedge(\frac{t}{s}\eta(z_i))\Big|n\in N, \sum_{i=1}^{n}y_iz_i=x\}$$
$$=((\frac{t}{s}\mu)\circ(\frac{t}{s}\eta))(x)$$
$$=(f(\mu)\circ f(\eta))(x),$$

that is $f(\mu \circ \eta)=f(\mu)\circ f(\eta)$, hence $(F_i^s(R), \circ)\cong(F_i^t(R), \circ)$.

If R has unit e, then $f(s_{\{e\}})=t_{\{e\}}$. □

4. The algebraic structure of the quotient set

In this section, we introduce some operations on the quotient set $F_i(R)/\sim$ by the operations of $F_i(R)$ and study their algebraic structure.

For any $\mu \in F_i(R)$, the symbol $[\mu]$ denotes the class which contains μ, i.e.,

$$[\mu]=\{\eta|\eta \in F_i(R) \text{ and } \eta(0)=\mu(0)\}.$$

Theorem 4.1 The following definitions are all the operations of quotient set $F_i(R)/\sim$.

(1) $[\mu]\vee[\eta]=[\mu\vee\eta]$,

(2) $[\mu]\wedge[\eta]=[\mu\wedge\eta]$,

(3) $[\mu]+[\eta]=[\mu]+[\eta]$,

(4) $[\mu]\circ[\eta]=[\mu\circ\eta$.

Proof. Omitted. □

Theorem 4.2 The quotient set $F_i(R)/\sim$ forms a complete lattice under the operation (1),(2) of Theorem 4.1, denoted by $(F_i(R)/\sim,\vee,\wedge)$.

Proof. We can easily obtain this result from Theorem 2.1 and the natural map from $F_i(R)$ to $F_i(R)/\sim$. □

Theorem4.3 $F_i(R)/\sim$ forms a monoid to the operation (3) of Theorem 4.1 with the unit $[1_{\{0\}}]$, denoted by $(F_i(R)/\sim,+)$, and

$$(F_i(R),+)\sim\ (F_i(R)/\sim,+).$$

Theorem 4.4 $(F_i(R)/\sim,+)$ forms a semigroup to the operation (4) of the Theorem 4.1, denoted by $(F_i(R)/\sim,\circ)$, and

$$(F_i(R),\circ)\sim(F_i(R)/\sim,\circ).$$

If R has unit, then $(F_i(R)/\sim,\circ)$ is monoid with unit $[1_{\{e\}}]$, and the homomorphism above preserve the unit.

The proof of Theorem 4.3 and 4.4 is similar to the Theorem 4.2.

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