The Boolean Agebra with Shell and Dangerous Signal Recognition Logic

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Abstract

In this paper, the concept and the theory of Boolean algbras with shell are estabished at first, where the new concepts of poset with subgreatest element and subleast element, Boolean type sublattice of a lattice, lattice with heats, and Boolean algebra with shell are proposed. Secondly, a new kind of nonclassical logic, which takes a Boolean algebra with shell as valuation lattice, and is called dangerous signal recognition logic, is established. Some results are obtained. Especially, in this new logic system, a impotant theorem is obtained that $K^{\#}$ –pretautologies just coincide with the C_2–tautologies in classical propositional calculus C_2 and so coincide with the theorems in C_2, i.e., $QT(K^{\#}) = T(C_2) = \Phi^{\vdash}$.

Key words: Fuzzy Logic; Dangerous signal recognition logic; Boolean algebra with shell; lattice with hearts; subgreatest element; subleast element; K–valuation lattice; Kleene–Dienes implication operator; Wang Guojun implication operator.

First, we establish the concept and the theory of Boolean algebras with shell.

Definition 1. Two distinct elements $1^{\#}$ and $0^{\#}$ of a partial ordering set $(P,\leqslant)$ are called the *subgreatest element* and the *subleast element* in P respectively, if P has greatest element 1 and least element 0, and the greatest element and the least element of the partial ordering subset $P-\{0,1\}$ are just $1^{\#}$ and $0^{\#}$ respectively.

Clearly, if a partial ordering set P has subgreatest element and subleast element, then P must have 4 elements at leat.

If a partial ordering set P has subgreatest element and subleast element, then they have uniqueness.

Proposition 2. Let $(P,\leqslant,\neg)$ be a partial ordering set with the subgreatest element $1^{\#}$ and the subleast element $0^{\#}$ and an order–reversing involution —, then

$$\neg 0=1, \qquad \neg 1=0,$$
$$\neg 0^{\#}=1^{\#}, \qquad \neg 1^{\#}=0^{\#}.$$

Lemma 3. Let $(L, \leqslant, \vee, \wedge, \neg)$ be a lattice with an order–reversing involution —, then de Morgan dual laws must hold in L, i.e., for every $a, b \in L$,

(1) $\neg(a\vee b)=\neg a\wedge\neg b$.

(2) $\neg(a\wedge b)=\neg a\vee\neg b$.

Definition 4. Let L be a lattice, L_0 be a sublattice of L. If L_0 is a Boolean lattice itseif, then L_0 is called a *Boolean type sublattice* of L.

It is clear that every Boolean sublattice of a lattice L must be a Boolean type sublattice of L, but a Boolean type sublattice of L needn't be a Boolean sublattice. Where a sublattice L^* of L is said to be a *Boolean sublattice* means that for every $a\in L^*$,

$$\neg a\vee a=1, \qquad \neg a\wedge b=0,$$

we notice that 1 and 0 are the greatest element and the least element of L respectively, but needn't be the greatest element and least element of L^* itself.

Definition 5. Let L be a lattice. L is called a *lattice with hearts* if the following six conditions are satisfied:

(1) L has six elements at least.

(2) The partial ordering $\leqslant$ of L is nonlinear ordering.

(3) L has the greatest element 1 and the least element 0.

(4) L has the subgreatest element $1^{\#}$and the subleast element $0^{\#}$.

(5) L has an order–reversing involution $\neg$.

(6) There exists some maximal Boolean type sublattice $L^{\#}$of L which has four elements at least, such that the subgreatest element $1^{\#}$and the subleast element $0^{\#}$ of L are just the greatest element and the least element of $L^{\#}$respectively, and the restriction onto $L^{\#}$of the order–reversing involution $\neg$ in L just coincids with the Boolean complement $'$ of $L^{\#}$itself.

The maximal Boolean type sublattice $L^{\#}$ is called a *heart* of L, and

$\tilde{L} = L - \bigcup_{\#} L^{\#}$ is called the *shell* of L.

Clearly, a lattice with hearts may have many hearts, but only one shell.

Definition 6. Suppose that L is a distributive lattice with hearts, if L has only one heart, then L is called a *Boolean algebra with shell.*

Theorem 7. A lattice L with unique heart $L^{\#}$is a Boolean algebra with shell if and only if the unique shell $\tilde{L}$ of L has only two elements, i.e., $\tilde{L} = \{0,1\}$.

Theorem 8. A lattice L with hearts is a Boolean algebra with shell if and only if $L-\tilde{L}$ is the isomorphic embedding image of some Boolean algebra into L.

Theorem 9. Let B be any Boolean algebra (needn't finite), and $+\infty$, $-\infty \notin$ B. Let

$$\hat{B} = B\bigcup\{-\infty, +\infty\}, \quad \neg -\infty = +\infty, \quad \neg +\infty = -\infty,$$

and for every $b \in B$, $-\infty < b < +\infty$. Then $\hat{B}$ forms a Boolean algebra with shell, and B is just the unique heart of $\hat{B}$, $+\infty$ and $-\infty$ are the greatest element and the least element of $\hat{B}$, the greatest element 1 and the least element 0 of B are just the subgeatest element and the subleast element of $\hat{B}$. We call $\hat{B}$ the *Boolean algebra with shell generated by Boolean algebra* B.

Theorem 10. A lattice L with hearts is a Boolean algebra with shell if and only if L is the Boolean algebra with shell generated by some Boolean algebra.

Proposition 11. Let L be a Boolean algebra with shell. Then for every $x \in L$,

$$\neg x \vee x \in \{1^{\#}, 1\},$$

$$\neg x \wedge x \in \{0^{\#}, 0\}.$$

Proposition 12. Let L be a Boolean algebra with shell, $L^{\#}$ be the unique heart of L, and $\hat{L}$ be the unique shell of L. Then

(1) If $x \in L^{\#}$, then $\neg x \vee x = 1^{\#}$, $\neg x \wedge x = 0^{\#}$.

(2) If $y \in \tilde{L}$, then $\neg y \vee y = 1$, $\neg y \wedge y = 0$.

Note 13. If we deal with the unique heart $L^{\#}$and the unique shell $\hat{L}$ of a Boolean algebra L with shall separately, as two Boolean type sublattices of L, then they obey the all operation laws of usual Boolean algebras.

Secondly, recall classical propositional calculus system $\mathbf{C}_2$.

Definition 14. Let S be a nonempty set, and F(S) be a free algebra of type ($\neg$,$\vee$, $\wedge$,) generated by S, where — be an unary operator, $\vee$, $\wedge$, and$\rightarrow$ be three binary operators. The elements of F(S) are called *propositions*, and the elements of S are called *atomic propositions*.

Definition 15. In the free algebra F(S), those propositions with form as following are called *axioms*:

(1) $A \rightarrow (B \rightarrow A)$.

(2) $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$.

(3) $(\neg A \rightarrow - B) \rightarrow (B \rightarrow A)$.

We denote the set of all axioms by A.

Definitions 16. In the free algebra F(S), consider a special operation $*$, which act upon these ordered pair of propositions form as $(A \rightarrow B, A)$, such that

$$(A \rightarrow B) * A = B.$$

The special operation $*$ is called *Modus Ponens*, or more briefly, *MP*.

Definition 17. Let $\Gamma \subset F(S)$.We denote

$$\Gamma^{\vdash} = \{ A \in F(S) \mid \Gamma \vdash A \},$$

where Γ—A means that there exists a sequence of propositions in F(S),

$$A_1, A_2, \cdots, A_n = A,$$

such that for every $i \leqslant n$, either $A_i \in \Gamma \bigcup A$, or there are $j<i$ and $k<i$ with

$$A_j * A_k = A_i.$$

$\Gamma \vdash A$ is called a *proof* from Γ to A. $\vdash$ is called the *syntax closure operator* on F(S). $\Gamma^{\vdash}$ is called the *syntax closure* of Γ.Those propositions in the syntax closure $\Phi^{\vdash}$ about empty subset Φ of F(S) are called *theorems*.

Definition 18. The tetrad

$$C_2 = (F(S), A, *, \vdash)$$

is called the *syntax of classical propositional calculus.*

Definition 19. A mapping v: $F(S) \rightarrow \{0,1\}$ is called a *valuation* of F(S), if

(1) For every $A \in F(S)$, $v(\neg A) = 1 - v(A)$.

(2) For every pair $A,B \in F(S)$, $v(A \rightarrow B) = 0$ if and only if $v(A) = 1$ and $v(B) = 0$.

We denote the set of all valuations of F(S) by Ω.

Definition 20. let $A \in F(S)$. If for every $v \in \Omega$, $v(A) = 1$, then A is called a *tautology*. We denote the set of all tautologies in F(S) by $T(C_2)$.

Definition 21. The triad

$$\tilde{C}_2 = (F(S), \Omega, T)$$

is called the *semantic of classical propositional calculus.*

Definition 22. The ordered pair

$$C_2 = (C_2, \tilde{C}_2)$$

is called *classical propositional calculus*, where C_2 is the syntax of C_2 , and $\tilde{C}_2$ is the semantic of C_2.

Theorem 23. (Reliability and Completeness Theorem)

$$T(C_2) = \Phi^{\vdash}.$$

Thirdly, we are now going to establish a new kind of nonclassical logic system $K^{\#}$, which takes a Boolean algebra with shell as valuation lattice.

Definition 24. let L be a Boolean algebra with shell. If we take Kleene–Dienes operator $R_{KD}: L \times L \rightarrow L$,

$$R_{KD}(a,b) = a \rightarrow b = \neg a \vee b$$

as the implication operator in L, then L is called a *K–valuation lattice.* A mapping v_k: F(S)$\rightarrow$L is called a *K–valuation,* if v_k is a homomorphism of type ($\neg$, $\vee,\wedge,\rightarrow$). We denote the set of all K–evaluations on F(S) by Ω_k.

Definition 25. Let $A \in F(S)$ and $\alpha \in L-\{0\}$.If for every K–valuation $v_k \in \Omega_k$, $v_k(A) \geqslant \alpha$ $v(_k(A) > \alpha$, $v_k(A) > 0$, $v_k(A) = 1)$, then A is called an *α–tautology (α^+–tautology, pretautology, tautology).* We denote the set of all α–tautologies (α^+ –tautologies, pretautologies, tautologies) by α–T($\mathbf{K}^{\#}$) (α^+ –T($\mathbf{K}^{\#}$), QT($\mathbf{K}^{\#}$),T($\mathbf{K}^{\#}$)).

Definition 26. The sextupe

$$\mathbf{K} = (F(S), \Omega_k, \alpha\text{–}T, \alpha^+\text{–}T, QT, T)$$

is called the *semantic of dangerous signal recognition logic* $\mathbf{K}^{\#}$.

Definition 27. The ordered pair $\mathbf{K}^{\#} = (C_2, \mathbf{K})$ is called a *dangerous signal recognition logic.*

Proposition 28. For all $\alpha, \beta \in L-\{0\}$,

$$(\alpha\text{–}T(\mathbf{K}^{\#})) \cap (\beta\text{–}T(\mathbf{K}^{\#})) = (\alpha \vee \beta)\text{–}T(\mathbf{K}^{\#}).$$

Note 29. In a K–valuation lattice L, the implication operater R_{KD} doesn't coincide with Wang Guojun operator R_0: $L \times L \rightarrow L$,

$$R_0(a,b) = \begin{cases} 1, & a \leqslant b, \\ \neg\ a \vee b, & a \nleqslant b. \end{cases}$$

because for every element c of the heart $L^{\#}$,

$$c \rightarrow c = \neg\ c \vee c = 1^{\#} \neq 1.$$

Proposition 30. In the heart $L^{\#}$of a K–valuation lattice L, the implication operator $\rightarrow$ is equivalent to Wang Guojun operator R_0:$L^{\#} \times L^{\#} \rightarrow L^{\#}$,

$$R_0(a,b) = \begin{cases} 1^{\#}, & a \leqslant b, \\ \neg a \vee b, & a \nleqslant b. \end{cases}$$

Theorem 31. in an arbitrary K–valuation lattice L, three binary operators $\vee$, $\wedge$, and$\rightarrow$ can be repressented each other as following:

(1) $a \vee b = \neg a \rightarrow b$.

(2) $a \vee b = \neg(\neg a \wedge \neg b)$.

(3) $a \wedge b = \neg(a \rightarrow \neg b)$.

(4) $a \wedge b = \neg(\neg a \vee \neg b)$.

(5) $a \rightarrow b = \neg a \vee b$.

(6) $a \rightarrow b = \neg(a \wedge \neg b)$.

Proposition 32. Any K–valuation lattice has minimal sufficient sets about connective words as following:

$$(\neg, \rightarrow), (\neg, \vee), (\neg, \wedge).$$

Proposition 33. In any K–valuation lattice, the unary operator $\neg$ may be repressented by the binary operator $\rightarrow$ as following:

$$\neg a = a \rightarrow 0.$$

Theorem 34. In an arbitrary K–valuation lattice, following equalities must hold:

(1) $b \wedge (a \rightarrow b) = b$.

(2) $b \vee (a \rightarrow b) = a \rightarrow b$.

(3) $\neg a \vee (a \rightarrow b) = a \rightarrow b$.

(4) $a \rightarrow (b \wedge c) = (a \rightarrow b) \wedge (a \rightarrow c)$.

(5) $a \rightarrow (a \rightarrow b) = a \rightarrow b$.

Proposition 35. In any K–valuation lattice,

$$b \vee (a \rightarrow b) = a \rightarrow (a \rightarrow b).$$

Note 36. A K–valuation lattice L needn't be a Heyting algebra, because that $a \rightarrow a = 1$ does not hold generally. For example, if $c \in L^{\#}$, then

$$c \rightarrow c = \neg c \vee c = 1^{\#} \neq 1.$$

Theorem 37. In any K–valuation lattice L, if $a \neq 1$, then

$$a \vee b = (a \rightarrow b) \rightarrow b.$$

Corollary 38. In any K–valuation lattice L, if $a \neq 1$ and $b \neq 1$, then

$$(a \rightarrow b) \rightarrow b = (b \rightarrow a) \rightarrow a.$$

Theorem 39. In any K–valuation lattice L, if $b \notin L^{\#}$, then

$$a \vee b = (a \rightarrow b) \rightarrow b.$$

Corollary 40. In any K–valuation lattice L, if $a, b \notin L^{\#}$, then

$$(a \rightarrow b) \rightarrow b = (b \rightarrow a) \rightarrow a.$$

α–HS rule [1] means that from $a \rightarrow b \geqslant \alpha$ and $b \rightarrow c \geqslant \alpha$ refer $a \rightarrow c \geqslant \alpha$.

Theorem 41. In any K–valuation lattice L,

(1) If $a \neq 1$, then 1–HS hold.

(2) If $c \neq 0$, then 1–HS hold.

(3) If $b \in \tilde{L}$, then 1–HS hold.

Theorem 42. In any K–valuation lattice L, assume $\alpha \neq 0$.

(1) If $a \neq 1$, then α–HS hold.

(2) If $c \neq 0$, then α–HS hold.

(3) If $b \in \tilde{L}$, then α HS hold.

α–MP rule [1] means that from $a \rightarrow b \geqslant \alpha$ and $a \geqslant \alpha$ refer $b \geqslant \alpha$.

Theorem 43. In any K–valuation lattice L, if $\alpha \in L-\{0^{\#}, 0\}$,then α–MP must hold.

Note 44. In any K–valuation lattice L, $0^{\#}$–MP does not hold. For example, take $a = 0^{\#}$and $b = 0$, then

$$a \rightarrow b = \neg a \vee b = 1^{\#} \vee 0 = 1^{\#} \geqslant 0^{\#},$$

but $b < 0^{\#}$.

Lemma45. Assume $\mathbf{K}^{\#}$is a dangerous signal recognition logic, F(S) is the free algebra of type $(\neg, \vee, \wedge, \rightarrow)$ generated by S in $\mathbf{K}^{\#}$,$S = \{p_1, p_2, \cdots\}$, $E \subset \{p_1, \cdots, p_n\}$,L is the K–valuation lattice in **K**,$\rightarrow$ is Kleene–Dienes implication operator in L. Make mapping φ: $S \rightarrow L$ such that $\varphi(S-E) = \{1\}$. Make mapping φ^*:$S \rightarrow \{0, 1\}$ such that

$$\varphi^*(p_i) = \begin{cases} \varphi(p_i), & if\ \varphi(p_i) \in \tilde{L}, \\ 1, & if\ \varphi(p_i) \in L^{\#}, \end{cases}$$

$$i = 1, \cdots, n.$$

Then there are a K–valuation v_φ induced by φ and a (0, 1)–valuation v_{φ^*} induced by φ^* such that

(1) If $v_\varphi(A) = 1$, then $v_{\varphi^*}(A) = 1$.

(2) If $v_{\varphi^*}(A) = 0$, then $v_{\varphi^*}(A) = 0$.

According to lemma 45, following important theorem is obtained at once.

Theorem 46. ($\mathbf{K}^{\#}$–Precompleteness Theorem) In an arbitrary dangerous signal recognition logic $\mathbf{K}^{\#}$, the pretautologies just coincide with the tautologies in classical propositional calculus $\mathbf{C}_2$ and so coincide with the theorems in $\mathbf{C}_2$, i.e.,

$$QT(\mathbf{K}^{\#}) = T(\mathbf{C}_2) = \Phi^{\vdash}.$$

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