Lattice implication algebras and MV—algebras

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Abstract Lattice implication algebras is an algebraic structure which is established by combining lattice and implication algebras. In this paper, the relationship between lattice implication algebras and MV—algebra was discussed, and then proved that both of the categorys of the two algebras are categorical equivalence. Finally, the infinitely distributivity in lattice implication algebras were proved.

Keywords lattice implication algebras, MV—algebras, lattice order groups, categorical equivalence

1. Introduction

In order to research the logical system whose propositional value is given in a lattice from the semantic viewpoint, we have proposed the concept of lattice implication algebras in [1], and have discussed their some properties. MV-algebras were invented by C.C.Chang[2]in order to provide an algebraic proof of the completeness theorm of the infinite-valued logic of Lukasiewicz. Having served this purpose, the properties of these algebras have been discussed by many people. In this paper, we discussed the relationship of the two algebraic structures. Futher, we obtain an important property of lattice implication algebras. i.e. the infinitely distributivity.

2. Preliminaries

Definition 1[1]. Let $(L, \vee, \wedge, ')$ be a complemented lattice with the universal bounds 0, I. $\rightarrow$: $L \times L$ be a mapping, $(L, \vee, \wedge, ', \rightarrow)$ is called a lattice implication algebra if it satisfies for all a, b, c $\in$ L, the following conditions:

(I_1) $a \rightarrow (b \rightarrow c) = b \rightarrow (a \rightarrow c)$

(I_2) $a \rightarrow a = I$

(I_3) $a \rightarrow b = b' \rightarrow a'$

(I_4) $a \rightarrow b = b \rightarrow a = I \Rightarrow a = b$

(I_5) $(a \rightarrow b) \rightarrow b = (b \rightarrow a) \rightarrow a$

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(k_1) $(a \vee b) \to c = (a \to c) \wedge (b \to c)$

(k_2) $(a \wedge b) \to c = (a \to c) \vee (b \to c)$

If "$\to$" satisfies $(I_1)-(I_5)$, then $(L, \vee, \wedge, ', \to)$ is said to be a quasi-lattice implication algebra.

Recall the properties of lattice implication algebra in [1], we give some needed results.

Proposition1[1]. In lattice implication algebra $L = (L, \vee, \wedge, ', \to, O, I)$, the lattice operation $\vee$ and $\wedge$ can be defined by "$\to$" as: $x \vee y = (x \to y) \to y$, $x \wedge y = (x' \vee y')'$, and $(L, \vee, \wedge, ', O, I)$ is a complemented distributive lattice with universal bounds 0, I, where the order realition is been defined by:

$$x \leqslant y \quad \text{iff} \quad x \to y = I$$

Proposition 2[1]. $(L, \vee, \wedge, ', \to)$ is a lattice implication algebra iff $(L, \vee, \wedge, ', \to)$ is a quasi-lattice implication algebra, and satisfies the following conditions:

(1) $(L, \vee, \wedge)$ is a distributive lattice;

(2) for all $x, y \in L$, $(x \to y) \to y = x \vee y$

(3) for all $x, y \in L$, $x \leqslant y$ iff $x \to y = I$

The concept about lattice implication hommorphism and lattice implication algebra category could be seen in [5-6].

Definition 2[2]. An MV-algebra is an algebra $(A, +, \cdot, *, O, I)$, where A is a nonempty set, 0 and I are constant elements of A, $+$ and $\cdot$ are binary operations, and $*$ is a unary operation, satisfying the following axioms (where we let $x \vee y = (x \cdot y^*) + y$, $x \wedge y = (x + y^*) \cdot y$):

Ax1	$x + y = y + x$	Ax1'	$x \cdot y = y \cdot x$
Ax2	$x + (y + z) = (x + y) + z$	Ax2'	$x \cdot (y \cdot z) = (x \cdot y) \cdot z$
Ax3	$x + x^* = I$	Ax3'	$x \cdot x^* = 0$
Ax4	$x + I = I$	Ax4'	$x \cdot 0 = 0$
Ax5	$x + 0 = x$	Ax5'	$x \cdot I = x$
Ax6	$(x + y)^* = x^* \cdot y^*$	Ax6'	$(x \cdot y)^* = x^* + y^*$
Ax7	$x = (x^*)^*$	Ax8	$0^* = I$

Ax9 $x \vee y = y \vee x$ Ax9' $x \wedge y = y \wedge x$

Ax10 $x \vee (y \vee z) = (x \vee y) \vee z$ Ax10' $x \wedge (y \wedge z) = (x \wedge y) \wedge z$

Ax11 $x + (y \wedge z) = (x + y) \wedge (x + z)$ Ax11' $x \cdot (y \vee z) = (x \cdot y) \vee (x \cdot z)$

Definition 3[2]. For all $x, y \in \mathbf{A}$, we write $x \leqslant y$ iff $x \vee y = y$.

Theorem3[2]. Let $(\mathbf{A}, +, \cdot, *, O, I)$ be an MV-algebra, then :

(1) The relation $\leqslant$ is a partial ordering over **A**; for all $x, y \in \mathbf{A}$, $x \vee y$ and $x \wedge y$ are respectively the sup and the inf of the pair (x, y) with respect to $\leqslant$; also, for every $x \in \mathbf{A}$, $0 \leqslant x \leqslant I$.

(2) **A** is a complemented distributive lattice with respect to the operations $\vee$, $\wedge$ and $*$.

Proof. (1) see [2 : proposition 1.11 , 1.14].

(2) The distributivity of A is proved in [9], and by [2 : Theoorem 1.2]: $(x \vee y)^* = x^* \wedge y^*$, and Ax7: $\forall x \in \mathbf{A}$, $x = (x^*)^*$, and from [7], $*$ is an inverse involution, so **A** is a complemented distributive lattice.

Definition 4[2]. Let $\mathbf{A}_1 = (\mathbf{A}_1, +, \cdot, *, O_1, I_1)$ and $\mathbf{A}_2 = (\mathbf{A}_2, +, \cdot, *, O_2, I_2)$ are the MV-algrbras, we say that the function Ψ: $\mathbf{A}_1 \rightarrow \mathbf{A}_2$ is a homomorphism from $\mathbf{A}_1$ to $\mathbf{A}_2$, If $\Psi(O_1) = O_2$, $\Psi(I_1) = I_2$, and Ψ preserves $+$, $\cdot$ and $*$ over **A**.

Definition 5[1]. The category of MV-algebra, denoted by **AC**, is the category has as objects all MV-algebra and as arrows all homomorphism between the corresponding MV-algebra.

3 . Lattice implication algebras and MV-algebras

In the following ,we will prove the categorical equivalence between the category of lattice implication algebras and the category of MV-algebras.

Given an MV-algebra $\mathbf{A} = (A, +, \cdot, *, O, I)$, we define $\Gamma(A) = (A, \rightarrow, ', O, I)$, by stipulating that for all $x, y \in \mathbf{A}$, $x \rightarrow y = x^* + y$ and $x' = x^*$; Further, given MV-algebra $\mathbf{A}_1$ and $\mathbf{A}_2$ and a homomophism Ψ : $\mathbf{A}_1 \rightarrow \mathbf{A}_2$, we define $\Gamma(\Psi) = \Gamma(\mathbf{A}_1) \rightarrow \Gamma(\mathbf{A}_2)$, by $\Gamma(\Psi) = \Psi$.

Theorem 1 . Γ is a categorical equivalence between the category of MV-algebras and the category of lattice implication algebras.

For the proof, we prepare a few lemmas:

Lemmas 1 . Γ(**A**) is a lattice implication algebra.

Proof. According to proposition 2 , we must prove that Γ(**A**) is a complemented lattice implication algebra, that is, Γ(A) satifies $(I_1) \sim (I_5)$, and satifies (1), (2) and (3) in proposition 2. In the following for every $x, y, z \in \mathbf{A}$,

concerning (I_1) $x \to (y \to z) = x^* + (y^* + z)$

$= (x^* + y^*) + z$ (Ax2)

$= (y^* + x^*) + z$ (Ax1)

$= y^* + (x^* + z)$ (Ax2)

$= y \to (x \to z)$

concerning (I_2) $x \to x = x^* + x = I$ (Ax3)

concerning(I_3) $x \to y = x^* + y = x^* + (y^*)^*$ (Ax7)

$= (y^*)^* + x^*$ (Ax1)

$= y^* \to x^*$

$= y' \to x'$

concerning(I_4), by [2 , Theorem 1.13], $\forall x, y \in \mathbf{A}$, $x \leqslant y$ iff $x^* + y = I$, then from the definition of Γ(A), $x \leqslant y$ iff $x \to y = I$. And because $x \to y = y \to x = I$, i.e $x^* + y = y^* + x = I$, hence $x \leqslant y$ and $y \leqslant x$, then $x = y$.

concerning (I_5) $(x \to y) \to y = (x^* + y)^* + y$

$= (x^{**} \cdot y)^* + y$ (Ax6)

$= (x \cdot y^*) + y$ (Ax7)

$= x \vee y$

$= y \vee x$ (Ax9)

$= y \cdot x^* + x$

$= (y^* + x^{**})^* + x$ (Ax6')

$= (y^* + x)^* + x$ (Ax7)

$= (y \to x) \to x$

During the proofs of (I_4) and (I_5), we have for all x, y $\in$ **A**, $x \leqslant y$ iff $x \to y = I$ and $(x \to y) \to y = x \vee y = (x \cdot y^*) + y$, and from proposition 3, **A** is a distributive lattice with respect to $\vee$ and $\wedge$, and * is a inverse involution, so A is a complemented distributive lattice with respect to $\vee$, $\wedge$ and *, hence (**A**, +, ·, *, O, I) and Γ(**A**) = (**A**, $\to$, ', O, I) both are complemented distributive lattice, using proposition 2, then Γ(**A**) is a lattice implication algebra.

Lemmas 2. Γ is a full and faithful functor from the category of MV-algebras and the category of lattice implication algebras.

Proof. It is easy to proved by the definition of Γ(A).

For the proof of Theorem 1, we provide an equivalent descripition of MV-algebra:

Lemmas 3[8]. Let **B** = (**B**, +, ·, *, O, I) be an algebra of type (2, 2, 1, 0, 0), then **B** is a MV-algebra iff **B** satisfies the following:

(p1) $(x + y) + z = x + (y + z)$ (p2) $x + 0 = x$

(p3) $x + y = y + x$ (p4) $x + I = I$

(p5) $(x^*)^* = x$ (p6) $O^* = I$

(p7) $x + x^* = I$ (p8) $(x^* + y)^* + y = (x + y^*)^* + x$

(p9) $x \cdot y = (x^* + y^*)^*$

Lemmas 4. Every lattice implication algebra **L** = (**L**, $\to$, ', O, I) is isomorphic (indeed equal) to Γ(A), for some MV-algebra **A**.

Proof. let **A** = (**L**, +, ·, *, O, I) be defined by stipulsting that for all x, y $\in$ **L**, we have : D1 $x^* = x'$; D2 $x + y = x' \to y$; D3 $x \cdot y = (x \to y')'$

To prove that **A** is a MV-algebra, by lemmas 3, it is sufficent to show that **A** obeys the equations (p1) ~ (p9).

concerning (p1) $(x + y) + z = (x' \to y)' \to z$ (by D2)

$= z' \to (x' \to y)$ (by I_3)

$= x' \to (z' \to y)$ (by I_1)

$= x' \to (y' \to z)$ (by I_3)

$= x + (y + z)$

concerning (p2) $x + o = x' \to o = x'' = x$ (" ' " is a inverse involution)

concerning (p3) $x + y = x' \to y = y' \to x = y + x$ (by D2 and I_3)

concerning (p4) $x + I = x' \to I = I' \to x = O \to x = I$ (proposition 2 in [2])

concerning (p5) $x^{**} = x'' = x$ (by D1)

concerning (p6) $o^* = o' = I$

concerning (p7) $x + x^* = x' \to x' = I$

concerning (p8) $(x^* + y)^* + y = (x \to y) \to y$ (by D2)

$= (y \to x) \to x$ (by I_1)

$= (y' + x)' + x$ (by D2)

$= (x + y^*)^{*} + x$ (by D1 and (p3))

concerning (p9) $x \cdot y = (x \to y')' = (x' + y')' = (x^* + y^*)^*$ (by D3 , D2 and D1)

We have just proved **A** is an MV-algebra. To complete the proof , we must prove $\Gamma(\mathbf{A}) = \mathbf{L}$. In fact, defining $x \to y = x^* + y$, $x' = x^*$, by Lemmas 1, knowing $\Gamma(\mathbf{A}) = (\mathbf{L}, \to, ', O, I)$ is a lattice implication algebra.

Further, for all $x, y \in \mathbf{L}$, $x \xrightarrow[(\Gamma(A))]{} y = x^* + y = (x^*)' \xrightarrow[(L)]{} y = (x')' \xrightarrow[(L)]{} y = x \xrightarrow[(L)]{} y$, hence, $\mathbf{L} = \Gamma(\mathbf{A})$.

Finally, by Lemmas 1 and Lemmas 4 , using the categorical equivalence theorem [4 , Theorem 1 , P91], we have: Γ is a categorical equivalence between the category of MV-algebras and the category of lattice implication algebras.

4 . The distributivity of lattice implication algebras

Following the above arguments, we can obtain some importment properties of lattice implication algebra.

Recall the realationship between MV-algebra and lattice-order abelian group.

Let $\mathbf{G} = (\mathbf{G}, +, -, 0, \vee, \wedge)$ be a lattice-order abelian group (called l-group in short) with order unit e[4], and we denote by $\leqslant_G$ the order induced on G by the lattice operations. Let $G^+=\{g \in G;\ g_G > 0\}$, An element $u \in G^+$ is a strong order unit in G ("order" unit , in the terminology of [10]) iff for each $g \in G$ there is a natural number n such that $g \leqslant_G nu$.

If (G , u) and (G', u') are l-group with order unit u and u' respectively , then a map $\lambda: G \to G'$ is said to be a unital l-homomorphism if and only if λ is a group homomorphism and a lattice homomorphism such that $\lambda(u) = u'$. Unital l-homomorphisms are precisely the morphisms in the category of l-group with order unit. We write $(G, u) \cong (G', u')$ if and only if there is an unital l-isomorphic from G onto G' .

Definition 6[3]. Let $G = (G, +, -, O_G, \wedge_G, \vee_G)$ be a l-group with order unit u. We define $\Sigma(G, u) = (A, \oplus, \cdot, *, O, I)$ by the following:

$A=[0, u]=\{g \in G;\ 0 \leqslant_G g \leqslant_G u\}$, and for all x , y ∈ **A** ,

$$x \oplus y = u \wedge_G (x + y),\quad x^* = u - x$$

$$x \cdot y = (x^* \oplus y^*)^* = 0 \vee_G (x + y - u),\quad O = O_G,\quad I = u$$

Further, given a unital l-homomorphism $\theta: (G, u) \to (G', u')$, we define $\Sigma(\theta): \Sigma(G, u) \to \Sigma(G', u')$ by $\Sigma(\theta) = \theta|_A$, that is $\Sigma(\theta)$ is a restriction of θ to **A** .

Recall from [3]:

Proposition 4[3]. Let Σ be the above definition, then Σ is a full and faithful functor from the category of l-group with order unit u to the category of MV-algebra. and for all these groups (G, u) the lattice group operations on the unit interval $[O_G, u]$ of the group G with order unit u, agree with the MV lattice operations on $A=\Sigma(G, u)$.

Proposition 5[3]. Let $A = (A, \oplus, \cdot, *, O, I)$ be an MV-algebra , then there exists an l-group $\Sigma(G, u)$ with unit u, such that $A \cong \Sigma(G, u)$.

As a n immediate consequence of proposition 4 and proposition 5, we have the following :

Theroem 2 . The functor Σ is an categorical equivalence between the category of l-groups with order unit , and the category of MV-algebras.

According to the above results, we are going to establish now the relationship between l-group with order unit and lattice implication algebra.

In the following, Γ is an equivalent functor from the category of MV-algebras to the category of lattice implication algebras as above, and Σ is an equivalent functor from the categorysof l-group with order unit to the category of lattice implication algebra as above.

Theorem 3 . The composite functor $\Omega = \Gamma \circ \Sigma$ of the functor Γ and Σ is a categorical equivalence from the category of l-groups with order unit to the category of lattice implication algebras.

Corollary 1 . For all lattice implication algebras, there must exist a l-group **G** with order unit u, such that **L** $\cong \Omega(G, u)$, where $\Omega(G, u)$ is $(G(u), \rightarrow, ', 0, I)$ by stipulating the following:

$G(u)=[0, u]=\{g \in G;\ O_G \leq_G g \leq_G u\}$, and for alll $x, y \in L$,

$x \rightarrow y = u \wedge_G (u - x + y), x' = u - x, I = u$.

Proof. We definite $x \oplus y = u \wedge_G (x + y)$, $x^* = u - x$, then $x^* + y = u \wedge_G (u - x + y)$. By the definitive approach of the functor Γ and Σ in theorem 3 , we have $\Omega=\Gamma \circ \Sigma$.

Using the above results, we will obtain the following :

We say that a lattice **L** $= < L, \wedge \vee >$ satisfies the infinitely distributive law provides the following property holds true:

If the family $\{x_i\}_{i \in J}$ of element of **L** has a supremum in **L**, then for each x in **L**, the family $\{x \wedge x_i\}_{i \in J}$ also has a supremum in **L** and we have :

$$x \wedge \left(\bigvee_{i \in J} x_i\right) = \bigvee_{i \in J} (x \wedge x_i)$$

It is well known (see, for instance, [10 , p312]) that each l-group satisfies the infinitely distributive law, since for each family $\{x_i\}_{i \in J}$ of elements of G(u), the supremum of $\{x_i\}_{i \in J}$ exists in G if and only if it exists in G(u). Then the infinitely distributive law holds in G[u]. By taking into account that isomorphic lattice implication algebras have isomorphic underlying lattice and by theorem 2, we have the following :

Theorem 4 . For all lattice implication algebra **L** $= (L, \vee, \wedge, ', \rightarrow, O, I)$, $(L, \vee, \wedge)$ satisfies the infinitely distributive law as above:

In lattice implication algebra $(\mathbf{L}, \vee, \wedge, \rightarrow, ')$, we have : for each $x, y, z \in \mathbf{L}$

$$x \rightarrow (y \vee z) = (x \rightarrow y) \vee (x \rightarrow y) \qquad (2)$$

By theorem 4 , we can extend $\vee$ in (2) to any one supremum.

Theorem 5. Let (L , ∨, ∧, ', →) be a lattice implication algebra, if the family $\{x_i\}_{i \in J}$ 在 of elements of L has a supremum in L, then we have :

$$x \to (\bigvee_{i\in J} x_i) = \bigvee_{i\in J} (x \to x_i) \qquad (3)$$

Proof. According to corollary 1, it is sufficent to verify that the equation (3) holds in G[u]. In this algebra, we have: $x \to (\bigvee_{i\in J} x_i) = u \wedge_G (u-x+(\bigvee_{i\in J} x_i))$, by the property of l-group in [10, P292]: $(u-x+(\bigvee_{i\in J} x_i)) = \bigvee_{i\in J} (u-x+x_i)$.

That the supremum in the left equation exists implies the existence of the supremum in the right equation, similarly, for the eqution $(x \vee y) \to z = (x \to y) \wedge (y \to z)$, we can replace the ∨ and ∧ by any supremum and any infimum.

In fact: $(\bigvee_{i\in J} x_i) \to b = \bigwedge_{i\in J} (x_i \to b)$ holds in residuated lattice, hence also holds in lattice implication algebras.

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