A NEW GENERALIZATION OF KY FAN'S FIXED POINT THEOREM FOR FUZZY MAPPINGS

SHI CHUAN and ZHU SHUNRONG

Department of Applied Mathematics
Nanjing University of Science & Technology
Nanjing, 210014, *People's Republic of China*

ABSTRACT: This paper bring forward a new generalization of KY Fan's fixed point theorem for fuzzy mappings, the results presented improve and generalize KY Fan's fixed point theorems and the corresponding recent important results.

KEY WORDS AND PHRASES: Fuzzy Mathematics, fuzzy mapping, fixed point theorem, KY Fan's fixed point theorem.

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1 PRELIMINARIES

Let X be a topological space, C be a nonempty subset of X, a mapping A: $C \to [0,1]$ is called a fuzzy subset over C, we denote by $\mathscr{F}(C)$ the famity of all fuzzy subsets over C, a maping F: $C \to \mathscr{F}(C)$ is called fuzzy mapping over C, let $A \in \mathscr{F}(C)$, $Q \in [0,1]$, set $(A)_Q = \{u \mid A(u) \geqslant Q, u \in C\}$ is called the Q−cut set of A.

DEFINITION 1.1 Let $F: C \to \mathscr{F}(C)$ be a fuzzy mapping, if $P \in C$ such that $Fp(p) = \max_{u \in C} Fp(u)$, then p is called a fixed point of F.

DEFINITION 1.2 Let $F: C \to \mathscr{F}(C)$ be a fuzzy mapping, if for any $\lambda \in [0,1]$ and any $y, z \in C$, it is true that

$$Fx(\lambda y + (1-\lambda)z) \geqslant \min\{Fx(y), Fx(z)\}$$

then F over C is called convex.

DEFINITION 1.3 Let $F:C \to \mathscr{F}(C)$ be a fuzzy mapping, if the membership function $Fx(y)$ is upper semi—continuous over $C \times C$ (as a real ordinary function), then F over C is called closed.

DEFINITION 1.4 Let $F:C \to \mathscr{F}(C)$, $O(x):C \to [0,1]$, $\forall\, x \in C$, $\widetilde{F}x = (Fx)_{O(x)} = \{u \mid Fx(u) \geqslant O(x),\ u \in C\} \subseteq C$, $\widetilde{F}:C \to 2^C$ is a set—valued mapping, $\forall\, D \subseteq C$, $\widetilde{F}(D) = \bigcup_{x \in D} \widetilde{F}x$, let $x_0 \in C$, $D_0 = \{x_0\}$, $D_1 = conv(\{x_0\} \cup \widetilde{F}x_0)$, $D_{n+1} = conv(\{x_0\} \cup \widetilde{F}(D_n))$, $n > 0$, $\because D_0 \subseteq D_1 \subseteq D_2 \subseteq \cdots \subseteq D_n \subseteq D_{n+1} \subseteq \cdots$, $\therefore V = \bigcup_{n \geqslant 0} D_n = \bigcup_{n \geqslant 0} D_{n+1} = \bigcup_{n \geqslant 0} conv(\{x_0\} \cup \widetilde{F}x_0(D_n)) = conv(\{x_0\} \cup \widetilde{F}(V))$, V is called the set induced by $\widetilde{F}$ and x_0.

LEMMA 1.5 (Ky Fan [1][3]) Let X be a real topological vector space which is locally convex and Hausdorff. Let C be a nonempty, compact and convex subset of X and $T:C \to 2^C$ a set—valued mapping having the following properties:

(1) for any x in C the set Tx is convex, compact and nonempty

(2) the set graph $T = \bigcup_{x \in C} \{(x,y),\ y \in Tx\}$ is a closed set in $X \times X$.

Then there exists a fixed point p of T, i.e. $p \in Tp$

LEMMA 1.6 Let X be a Hausdorff topological space, C be a nonempty, compact and convex subset of X, $T:C \to 2^C$ be a set—valued mapping such that $\forall\, x \in C$, set Tx is compact, if $T:C \to 2^C$ is upper semi—continuous, then set graph $T = \bigcup_{x \in C} \{(x,y),\ y \in Tx\}$ is a closed set in $X \times X$.

Proof. Let $x_K \to x_0, y_K \in Tx_K$, $y_K \to y_0$, we shall show that $y_0 \in Tx_0$. If $y_0 \notin Tx_0$, by X be a Hausdorff topological space, $\forall\, y \in Tx_0$, there exists a neighbourhood of y $Wy \subseteq X$, & a neighbourhood of y_0 $Vy \subseteq X$, $Wy \cap Vy = \varnothing$, $Tx_0 \subseteq \bigcup_{y \in Tx_0} \{Wy \mid y \in Tx_0\}$, by set Tx_0 is compact, there exits $y_1, y_2, \cdots, y_n \in Tx_0$, such that $Tx_0 \subseteq \bigcup_{i=1}^{n} Wy_i = W$. Let $V = \bigcap_{i=1}^{n} Vy_i$,

V is a neighbourhood of y_0, such that $W \cap V = \varnothing$, by $T: C \to 2^C$ is upper semi—continuous, there exists a neighbourhood of x_0 U such that $\forall\ x_K \in U$, $Tx_K \subset W$, thus $y_K \in W$, moreover $W \cap V = \varnothing$, $y_K \to y_0$, this is a contradiction, thus $y_0 \in Tx_0$. By $y_0 \in Tx_0$, thus graph T is a closed.

2 MAIN RESULTS

THEOREM 2.1 Let X be a real topological vector space which is locally convex and Hausdorff, C is a nonempty closed convex subset of X, $F: C \to \mathscr{F}(C)$ is a closed convex fuzzy mapping over C.

(1) Suppose that there exists a lower semi—continuous function $O(x)$: $C \to (0,1]$ such that $\forall\ x \in C$, $\widetilde{F}x = (Fx)_{O(x)} \neq \varnothing$, there exists $x_0 \in C$ such that the set V induced by $\widetilde{F}$ and x_0 is a compact subset of C, then there exits $p \in C$ such that $Fp(p) \geqslant O(p)$;

(2) If $O(x) = \max\limits_{u \in C} Fx(u)$: $C \to (0,1]$ such that $\forall\ x \in C$, $\widetilde{F}x = (Fx)_{O(x)} \neq \varnothing$, moveover for any $y \in C$, $Fx(y)$ as a function of $x \in C$ is lower semi—continuous, there exists $x_0 \in C$ such that set V induced by $\widetilde{F}$ and x_0 is a compct subset of C, then there exists a point $p \in C$ such that $Fp(p) = \max\limits_{u \in C} Fp(u)$, i. e. p is a fixed point of F. .

Proof. Let $\widetilde{F}x = (Fx)_{O(x)} = \{u \mid Fx(u) \geqslant O(x), u \in C\}$, then $\widetilde{F}: C \to 2^C$, the set V induced by $\widetilde{F}$ and x_0 is a compact subset of C, $V = conv(\{x_0\} \cup \widetilde{F}(V))$ $\forall\ x \in V$, $\widetilde{F}x \subseteq V$. First, we prove that for each $x \in V$, $\widetilde{F}x$ is a nonempty convex compact set of V. .

$\forall\ x \in V \subseteq C$, $\widetilde{F}x = (Fx)_{O(x)} \neq \varnothing$, if $\forall\ y, z \in \widetilde{F}x$, $\lambda \in [0,1]$ by $F: C \to \mathscr{F}(c)$ is a convex fuzzy mapping over C, $Fx(\lambda y + (1-\lambda)z) \geqslant \min\{F_x(y), Fx(z)\} \geqslant O(x)$, $\therefore \lambda y + (1-\lambda)z \in (Fx)_{O(x)} = \widetilde{F}x$, $\widetilde{F}x$ is a convex set. Let $\{y_j\}_{j \in I} \subseteq \widetilde{F}x$ and $y_j \to y_0 \in V \subseteq C$, thus $(x, y_j) \to (x, y_0)$ moreover $Fx(y_j) \geqslant O(x)$ by $F: C \to \mathscr{F}(C)$ is a closed fuzzy mapping, $Fx(y)$, as a function over $C \times C$, is upper semi—continuous. This leads to the conclusion that $Fx(y_0) \geqslant \overline{lim}_{j \in I} Fx(y_j) \geqslant O(x)$, $\therefore y_0 \in (Fx)_{O(x)} = \widetilde{F}x$, $\widetilde{F}x$ is a closed subset of V, by $\widetilde{F}x \subseteq V$, V is compact set,

$\therefore \widetilde{F}x$ is a compact set.

Next, we prove that set graph $\widetilde{F} = \bigcup_{x \in V} \{(x,y) \mid y \in \widetilde{F}x\}$ is a closed subset in $X \times X$. Let $\{(x_j, y_j)\}_{j \in I}$ is a net of graph $\widetilde{F}$, and $x_j \to x_0 \in V$, $y_j \to y_0 \in V$, by F is closed and $O(x): C \to (0,1]$ is lower semi-continuous, we have $Fx_0(y_0) \geqslant \overline{\lim_j} Fx_j(y_j) \geqslant \overline{\lim_j} O(x_j) \geqslant \underline{\lim} O(x_j) \geqslant O(x_0)$, $\therefore y_0 \in (Fx_0)_{O(x_0)} = \widetilde{F}x_0$, $(x_0, y_0) \in$ graph$\widetilde{F}$, graph $\widetilde{F}$ is closed in $X \times X$.

For $\widetilde{F}: V \to 2^V$ applying lemma 1. 5 (Ky Fan [1],[3]) there exists $p \in C$ such that $p \in \widetilde{F}p = (Fp)_{O(p)}$, i. e. $Fp(p) \geqslant O(p)$.

When $O(x) = \max_{u \in C} Fx(u): C \to (0,1]$ such that the conditions in (2). , by the same way we can prove that for $\forall\ x \in V$, $\widetilde{F}x$ is a nonempty convex compact set, now we prove the set graph $\widetilde{F}$ is closed in $X \times X$. Let $\{(x_j, y_j)\}_{j \in I}$ is a net of graph $\widetilde{F}$, $y_j \in \widetilde{F}x_j$, and $x_j \to x_0 \in V$, $y_j \to y_0 \in V$, $(x_j, y_j) \to (x_0, y_0) \in V \times V$, by F is closed and for any $y \in C$, $Fx(y)$ as a function of $x \in C$ is lower semi-continuous, we have: $Fx_0(y_0) \geqslant \overline{\lim_j} Fx_j(y_j) \geqslant \overline{\lim_j} O(x_j) \geqslant \overline{\lim_j} \max_{u \in c} Fx_j(u) \geqslant \overline{\lim_j} Fx_j(u) \geqslant \underline{\lim} Fx_j(u) \geqslant Fx_0(u)$, $\forall\ u \in C$, thus $Fx_0(y_0) \geqslant \max_{u \in C} Fx_0(u) = O(x_0)$, $y_0 \in \widetilde{F}x_0$, therefore $(x_0, y_0) \in$ grap$\widetilde{F}$, grap$\widetilde{F}$ is closed in $X \times X$. For $\widetilde{F}: V \to 2^V$ applying lemma 1. 5. there exits $p \in C$ such that $p \in \widetilde{F}p = (Fp)_{O(p)}$, i. e. $Fp(p) = \max_{u \in C} Fp(u)$, P is a fixed point of F. This completes the proof of Theorem 2. 1.

Let $T: C \to 2^C$ is a set-valued mapping, by using T we define a fuzzy mapping F as follows: $F: C \to \mathscr{F}(C)$, $x \to \mathscr{X}_{Tx}$, where $\mathscr{X}_{Tx}$ is the characteritic function of Tx, Taking $O(x) \equiv 1$, $\forall\ x \in C$ it is easy to prove that when $\forall\ x \in C$, Tx is a convex closed subset of C, $F: C \to \mathscr{F}(C)$ is a convex closed fuzzy mapping. Thus we have:

COROLLARY 2. 2 Let X, C satisfy the conditios of Theorem 2. 1. Let $T: C \to 2^C$ is a set-valued mapping such that $\forall\ x \in C$, Tx is a nonempty convex closed subset of C, and there exists $x_0 \in C$ the set V induced by T and x_0 is a compact subset of C. If set graph T is closed in X

$\times X$, then there exists $p \in C$ such that $p \in Tp$.

COROLLARY 2. 3 Let X, C, T satisfy the conditions of corollary 2. 1. If $T: C \to 2^C$ is upper semi—continuous, then there exists $p \in C$ such that $p \in Tp$.

Proof for $T: C \to 2^C$ applying lemma 1. 6, by $T: C \to 2^C$ is upper semi—continuous, we have set graph T is closed in $X \times X$.

REMARK by corollary 2. 2 and 2. 3, it is easy to see that Theorem 2. 1 improves and generalizes Ky Fan's fixed point and corresponding important results of [1,2,3,4,5]

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