Fuzzy Factor Algebras

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Abstract: In this paper, we will define the concept of fuzzy factor algebras, and prove the isomorphism theorem of fuzzy algebras.

Keywords: fuzzy algebra; fuzzy ideal, isomorphism.

1. Preliminaries

Let X be any set, A fuzzy set A in X is characterized by a mapping A: $X \rightarrow [0,1]$.

Definition 1.1. Let Y be an algebra over field K, and A a fuzzy subset of Y. The A is called a fuzzy algebra of Y if

1) $A(\lambda_1 x + \lambda_2 y) > \inf\{A(x), A(y)\}$

2) $A(xy) > \inf\{A(x), A(y)\}$

for all $x, y \in Y$ and $\lambda_1, \lambda_2 \in K$.

In following, if we speak of algebra Y, we always mean the algebra over field K.

Definition 1.2. Let B be a fuzzy algebra of algebra Y, if $B(xy) > B(x) \vee B(y)$, for all $x, y \in Y$, then B is called a fuzzy ideal of Y.

Definition 1.3. Let B be a fuzzy ideal of Y, the fuzzy subset $x + B$

of Y is defined as follows: $(x+B)y=B(x-y)$ for all $y \in Y$.

Definition 1.4. Let B be a fuzzy ideal of Y, $Y/B=\{x+B \mid x \in Y\}$. The operation "+", "·" and scalar product on B/I are defined as follows:

$$(x+B)+(y+B)=x+y+B$$
$$(x+B)(y+B)=xy+B$$
$$\lambda(x+B)=\lambda x+B$$

We can easily prove the Definition 1.4 to be fine definitions.

Proposition 1.5. Let B be a fuzzy ideal of algebra Y, then B/ I is an algebra over field K.

Proof. The proof is very easy, and hence omitted.

Proposition 1.6. Let B be a fuzzy ideal of the algebra Y, and $G_B=\{x \mid x \in Y,\ B(x)=B(0)\}$, then G_B is an ideal of Y, and $Y/G_B \cong Y/B$.

2. Fuzzy factor algebras

Let Y be an algebra over field K, A a fuzzy subalgebra of Y, B a fuzzy ideal of Y. We define a fuzzy set A/B of Y/B as follows:

$$A/B:\ Y/B \to [0,1]$$

and

$$A/B(x+B)=\sup_{y+B=x+B} A(y)$$

Proposition 2.1. A/B is a fuzzy subalgebra of Y/B.

Proof. For all $x, y \in Y$, $\lambda_1, \lambda_2 \in K$, we have

$$A/B(\lambda_1(x+B)+\lambda_2(y+B)) = A/B(\lambda_1 x+\lambda_2 y+B)$$

$$= \sup_{z=\lambda_1 x+\lambda_2 y} A(z) \geqslant \sup_{\substack{z_1=\lambda_1 x\\ z_2=\lambda_2 y}} A(z_1+z_2)$$

$$\geqslant \sup_{\substack{z_1=\lambda_1 X\\ z_2=\lambda_2 y}} \inf(A(z_1),A(z_2))$$

$$= \inf\{\sup_{z_1=\lambda_1 X} A(z_1), \sup_{z_2=\lambda_2 y} A(z_2)\}$$

$$\geqslant \inf\{A(\lambda_1 x), A(\lambda_2 y)\}$$

$$\geqslant \inf\{A(x), A(y)\}$$

$$A/B\ (x+B)(y+B) = A/B(xy+B)$$

$$= \sup_{z+B=xy+B} A(z)$$

$$\geqslant \sup_{\substack{x_1+B=x+B\\ y_1+B=y+B}} A(x_1 y_1)$$

$$\geqslant \sup_{\substack{x_1+B=x+B\\ y_1+B=y+B}} \inf\{A(x_1), A(y_1)\}$$

$$= \inf\{\sup_{x_1+B=x+B} A(x_1), \sup_{y_1+B=y+B} A(y_1)\}$$

$$= \inf\{A/B(x+B), A/B(y+B)\}$$

Hence, A/B is a fuzzy subalgebra of Y/B.

Definition2.2. We call A/B the fuzzy factor algebra of A about B.

Definition 2.3. Let Y, Y′ be general sets, $f: Y \rightarrow Y'$ a suriective mapping, and A a fuzzy set of Y. If $f(x)=f(y)$ follows $A(x)=A(y)$, then A is called f-invariant.

Definition 2.4. Let $f: Y \rightarrow Y'$ be an algebra homomorphism (isomorphism), A and A′ fuzzy algebral of Y and Y′, respectively If $f(A)=A'$, then we say A is homomorphic (isomorphic) to A′, which is denoted as $A \sim A'$ ($A \cong A'$).

Definition 2.5. Let Y, A and B be as above, then $A \sim A/B$.

Proof Let $g: A \rightarrow A/B$ as $g(x) = x + B$ for all $x \in Y$. Then we have

$$g(\lambda_1 x + \lambda_2 y) = \lambda_1 x + \lambda_2 y + B = \lambda_1(x+B) + \lambda_2(y+B)$$
$$= \lambda_1 g(x) + \lambda_2 g(y)$$
$$g(xy) = xy + B = (x+B)(y+B) = g(x)g(y)$$

for all $x, y \in Y$. Thus g is a ring homomorphism:

$$A/B(x+B) = \sup_{y+B=x+B} A(y) = \sup_{B(y)=x+B} A(y) = g(A)(x+B)$$

Hence $A \sim A/B$.

Here g is called the natural homomorphism.

Proposition 2.6. Let f be an algebra homomorphism from algebra Y onto algebra Y', A the fuzzy subalgebra of Y and I the ideal of Y. If $G_B \subset \ker f$, then $A/B \sim f(A)$.

Proof. Let $g: Y/B \rightarrow Y'$, $g(x+B) = f(x)$, for all $x+B \in Y/B$. If $x \neq y$, $x+B = y+B$, then $B(x-y) = B(0)$, $x-y \in G_B \subset \ker f$, $f(x) = f(x-y+y) = f(x-y) + f(y) = f(y)$. Thus g is a mapping.

For all $x+B$, $y+B \in Y/B$, λ_1, $\lambda_2 \in F$

$$\begin{aligned} g\ (\lambda_1(x+B) + \lambda_2(y+B)) &= g(\lambda_1 x + \lambda_2 y + B) \\ &= f(\lambda_1 x + \lambda_2 y) \\ &= \lambda_1 f(x) + \lambda_2 f(y) \\ &= \lambda_1 g(x+B) + \lambda_2 g(y+B) \end{aligned}$$

$$\begin{aligned} g\ ((x+B)(y+B)) &= g(xy+B) \\ &= f(xy) \\ &= f(x)f(y) \\ &= g(x+B)g(y+B) \end{aligned}$$

Thus, g is an algebra homomorphism. For all $x' \in Y'$,

$$g(A/B)(x') = \sup_{g(x+B)=x'} A/B(x) = \sup_{f(x)=x'} \sup_{y+B=x+B} A(y)$$
$$= \sup_{f(y)=x'} A(y) = f(A)(x')$$

Hence $A/B \sim f(A)$.

By proposition 1.6, 2.5, 2.6 we have the following theorem.

Theorem 2.7. Let $f: Y \to Y'$ be an algebra homomorphism, A a fuzzy algebra of Y, B a fuzzy ideal of Y and $G_B = \ker f$; then $A/B \cong f(A)$.

Proposition 2.8. Let $f: Y \to Y'$ be an algebra homomorphism, B a fuzzy ideal of Y and B be f-invariant. Then $Y/B \cong Y'/f(B)$.

Proof Let $g: Y/B \to Y'/f(B)$ and $g(x+B) = f(x) + f(B)$

we can easily prove that g is isomorphism from Y/B onto $Y'/f(B)$ so $Y/B \cong Y'/f(B)$.

Theorem 2.9 Let $f: Y \to Y'$ be an algebra homomorphism, A a fuzzy algebra of Y and B a fuzzy ideal of Y. If B is f-invariant, then $A/B \cong f(A)/f(B)$.

Proof. We have the following diagram

$$Y \to Y' \to Y'/f(B)$$
$$A \to f(A) \to f(A)/f(B)$$

It is clear $f(A)/f(B) = g(f(A))$, and g is the natural homomorphism. By Theorem 2.7. We need only to prove $\ker(gf) = G_B$. It is clear $B(0) = f(B)(0')$ For all $x \in G_B$, $B(x) = B(0)$:

$$f(B)(f(x)) = \sup_{f(y)=f(x)} B(y) = B(0) = f(B)(0')$$

Hence, $f(x)f(B) = f(B)$, $x \in \ker(gf)$. For all $x \in \ker(gf)$, $(gf)(x) = f(B)$, $f(x)f(B) = f(B)$, $f(B)(f(x)) = f(B)(0) = B(0)$, $\sup_{f(y)=f(x)} B(y) = B(0)$.

Since B is f-invariant, we have $f(x) = f(y)$ following $B(x) = B(y)$. Hence $B(x) = B(0)$, $x \in G_B$. Thus $G_B = \ker(gf)$.

Reference

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