THE IMPORTANT PROPERTIES OF FUZZY ALGEBRA OVER FUZZY FIELDS

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ABSTRACT

We'll discuss in this paper, decomposition theorem 、 characteristic depiction and some other important properties of fuzzy algebra over fuzzy fields.

Keywords: Fuzzy fields, fuzzy algebra over fuzzy fields, decomposition theorem, characteristic depiction

1 PREEELIMINARY

Let X be any not empty set and let L be a complete distributive lattice in which there could be 0 and 1. Then a fuzzy subset A to X is characterized by a map $A:X\to L$, $A_\alpha=\{x\in X\mid A(x)\geq\alpha\}$, here $\alpha\in L$, A_α could be a α—level cut of A. When α 、 $\beta\in L$, if $\alpha\leq\beta$,then $A_\alpha\geq A_\beta$. F could be a fuzzy subset of X, F_α is a α—level cut of F as well. Let X be an ordinary field , in this paper, F refers to the fuzzy field of X, Y refers to algebra over field X.

In the following passages we will define the fuzzy algebra over fuzzy field.

Definition 1.1 Let Y be algebra over field X and let A be a fuzzy subset of Y, F be a fuzzy field of X, if $\forall$ x, $y\in Y$ and $\lambda\in X$,

(i) $A(x+y)\geq A(x)\wedge A(y)$;

(ii) $A(x\cdot y)\geq A(x)\wedge A(y)$;

(iii) $A(\lambda x)\geq F(\lambda)\wedge A(x)$;

(iv) $A(x)\geq A(-x)$;

(v) $A(0)=1$.

Then A is could a fuzzy algebra over fuzzy field F of Y.

Due to the definition 1.1,we can easily draw the following conclusions.

Proposition 1.1 Suppose A is a fuzzy algebra over fuzzy field F of Y, then the following conclusions hold.

(i) $\forall$ $x\in Y, A(x)=A(-x)$;

(ii) $\forall$ x, $y\in Y, A(x-y)\geq A(x)\wedge A(y)$;

(iii) $\forall$ x, $y\in Y$, if $A(x)\leq A(y)$,then $A(x+y)\wedge A(y)=A(x)$;

(iv) $\forall x,y\in Y$, if $A(x)\leq A(y)$,then $A(x\cdot y)\wedge A(y)=A(x)$;

(v) $\forall$ $x\in Y$ and $\lambda\in X$, if $A(x)\leq F(\lambda)$,then $A(\lambda x)\wedge A(x)=A(x)$.

In fact, (i) and (iv) in definition 1.1 is equal to (ii) in Proposition 1.1.

2 IMPORTANT PROPERTIES

According to the former definition and proposition, we prove decomposition theorem, characteristic and some other important properties of fuzzy algebra over fuzzy fields.

Theorem 2.1 Let F be a subset of X, and A be a fuzzy subset of Y, then A is a fuzzy algebra over fuzzy field F of Y iif $\forall \alpha \in L$, A_α is a subspace of Y, in particular when F_α is not empty, A_α is an algebra over field F_α.

Proof. Necessity: $\forall \alpha \in L$, due to the fuzzy algebra and the fuzzy space theory, easily let A_α be a subspace of Y and F_α be a subfield of X, when F_α is not empty. Then $\forall\ x, y \in A_\alpha, \lambda \in F_\alpha, A(x) \geq \alpha, A(y) \geq \alpha, F(\lambda) \geq \alpha$. Because A is a fuzzy algebra over fuzzy field F of Y, then

(i) $A(x+y) \geq A(x) \wedge A(y) \geq \alpha$;

(ii) $A(x \cdot y) \geq A(x) \wedge A(y) \geq \alpha$;

(iii) $A(\lambda x) \geq F(\lambda) \wedge A(x) \geq \alpha$.

Therefore A_α is an algebra over field F_α.

Sufficiency: $\forall\ x,\ y \in Y, \lambda \in X$, Let $\alpha = A(x) \wedge A(y)$, $\beta = F(\lambda) \wedge A(x)$, due to A_α and A_β are both subspaces of Y. Respectfully, A_α and A_β are algebras over fields F_α and F_β, when F_α and F_β are not empty. Therefore, $x+y \in A_\alpha, x \cdot y \in A_\alpha, \lambda x \in A_\beta$, and

(i) $A(x+y) \geq \alpha = A(x) \wedge A(y)$;

(ii) $A(x \cdot y) \geq \alpha = A(x) \wedge A(y)$;

(iii) $A(\lambda x) \geq \beta = F(\lambda) \wedge A(x)$,

Easily to prove: $A(0)=1, A(x)=A(-x)$, therefore A is a fuzzy algebra over fuzzy field F of Y.

Theorem 2.2 Let L be a complete distributive lattice, fuzzy subset A of Y is fuzzy algebra over fuzzy field F iif there is a subfield family $\{X_\alpha | \alpha \in L,\ X_\alpha$ is a subfield of $X, \bigcap_{\substack{\alpha \in H \\ H \subseteq L}} X_\alpha \subseteq X_{\text{supp}H}\}$, and a subalgebra family $\{Y_\alpha | \alpha \in L, Y_\alpha$ is a subset of $Y, \bigcap_{\substack{\alpha \in H \\ H \subseteq L}} Y_\alpha \subseteq Y_{\text{supp}H}\}$, such that Y_α is a subspace of Y. In particular, when $\alpha \leq \sup_{\alpha \in L} p\ F(\lambda)$, Y_α is an algebra over field X_α, and $F = \bigcup_{\alpha \in L} \alpha \cdot \tilde{X}_\alpha$, $A = \bigcup_{\alpha \in L} \alpha \cdot \tilde{Y}_\alpha$, $\tilde{X}_\alpha$ and $\tilde{Y}_\alpha$ respectively indicate characteristic functions of X_α and Y_α.

Proof. Necessity: Due to decomposition theorem of fuzzy subset, we can conclude $F = \bigcup_{\alpha \in L} \alpha \cdot \tilde{F}_\alpha$, $A = \bigcup_{\alpha \in L} \alpha \cdot \tilde{A}_\alpha$. Here $\tilde{F}_\alpha$ and $\tilde{A}_\alpha$ respectively indicate characteristic functions of F_α and A_α, due to definition 1.1, we have: $\forall \alpha \in [0, \text{supp} F(\lambda)]$, A_α is a subspace of Y, and A_α is an algebra over field F_α when F_α is not empty. We easily obtain: $\bigcap_{\substack{\alpha \in H \\ H \subseteq L}} X_\alpha = X_{\text{supp}H}$, $\bigcap_{\substack{\alpha \in H \\ H \subseteq L}} Y_\alpha \subseteq Y_{\text{supp}H}$. Here $X_\alpha = F_\alpha$, $Y_\alpha = A_\alpha$, now necessity holds.

Sufficiency: Let $F=\bigcup_{\alpha\in L}\alpha\cdot X_\alpha$, $A=\bigcup_{\alpha\in L}\alpha\cdot\tilde{Y}_\alpha$, and A_α is a subspace of Y, when $\alpha\in L$, in particular, when $\alpha\leq\sup_{\alpha\in L} p\ F(\lambda)$, Y_α is an algebra over field X_α. $\forall$ $x\in Y$, if x is not any a subset family $\{Y_\alpha|\alpha\in L\}$, then $A(x)=0$. If $x\in Y_\alpha$, $A(x)=\bigcup_{x\in Y_\alpha}\alpha$, therefore $A(x)=\bigcup_{x\in Y_\alpha}\alpha$ when supposed empty set's supreme is 0. So $\forall$ x, $y\in Y$, if $A(x)=\bigcup_{x\in Y_\alpha}\alpha=\mu$, $A(y)=\bigcup_{y\in Y_\beta}\beta=\nu$, then $x\in\bigcap_{x\in Y_\alpha}Y_\alpha=Y_{\vee_{x\in Y_\alpha}\alpha}=Y_\mu$, $y\in\bigcap_{y\in Y_\beta}Y_\beta=Y_{\vee_{y\in Y_\beta}\beta}=Y_\nu$.

Because $Y_\mu\cap Y_{\mu\wedge\nu}=Y_\mu$, $Y_\nu\cap Y_{\mu\wedge\nu}=Y_\nu$, $Y_\nu\subseteq Y_{\mu\wedge\nu}$, $Y_\mu\subseteq Y_{\mu\wedge\nu}$, thus $\forall x,y\in Y_{\mu\wedge\nu}$, have $x-y\in Y_{\mu\wedge\nu}$, so we have $A(x-y)=\vee[\alpha\wedge\tilde{Y}_\alpha(x-y)]\geq(\mu\wedge\nu)\wedge Y_{\mu\wedge\nu}(x-y)=\mu\wedge\nu$, that is $A(x-y)\geq A(x)\wedge A(y)$; $A(x\cdot y)=\vee[\alpha\wedge\tilde{Y}_\alpha(x\cdot y)]\geq(\mu\wedge\nu)\wedge Y_{\mu\wedge\nu}(x\cdot y)=\mu\wedge\nu$, that is $A(x\cdot y)\geq A(x)\wedge A(y)$.

Secondly, $\forall$ $x\in Y,\lambda\in X$, if $F(\lambda)=\bigcup_{\lambda\in X_\alpha}\alpha$, $A(x)=\bigcup_{x\in Y_\beta}\beta$, let $F(\lambda)=\delta$, $A(x)=\varepsilon$. Then because $\lambda\in\bigcap_{\lambda\in X\alpha}X_\alpha=X_{\vee_{\lambda\in X\alpha}\alpha}=X_\delta$, $x\in\bigcap_{x\in Y_\beta}Y_\beta=Y_{\vee_{y\in Y_\beta}\beta}=Y_\varepsilon$. Because $X_\delta\subseteq X_{\delta\wedge\varepsilon}$, $Y_\varepsilon\subseteq Y_{\delta\wedge\varepsilon}$, but $Y_{\delta\wedge\varepsilon}$ is an algebra of $X_{\delta\wedge\varepsilon}$, hence $\lambda x\in Y_{\delta\wedge\varepsilon}$ and $A(\lambda x)=\vee[\alpha\wedge\tilde{Y}_\alpha(\lambda x)]\geq(\delta\wedge\varepsilon)\wedge Y_{\delta\wedge\varepsilon}(\lambda x)=\delta\wedge\varepsilon$, that is $A(\lambda x)\geq F(\lambda)\wedge A(x)$.

We also have $A(0)=\bigcup_{\alpha\in L}\alpha=1$, so A is a fuzzy algebra over fuzzy field F of Y.

That is sufficiency holds.

Definition 2.1. Let Y be an algebra over field X, F be fuzzy field of X, A and B are fuzzy subsets of $Y,\lambda\in X$, define $A\cap B, A+B, A\cdot B,\lambda A,-A$ respectively is the following fuzzy subset of Y. $\forall$ $x\in Y$

$(A\cap B)(x)=A(x)\wedge B(x)$; $(A+B)(x)=\bigcup_{x_1+x_2=x}[A(x_1)\wedge B(x_2)]$;

$(A\cdot B)(x)=\bigcup_{x_1\cdot x_2=x}[A(x_1)\wedge B(x_2)]$; $(\lambda A)(x)=\bigcup_{\lambda x_1=x}[F(\lambda)\wedge A(x_1)]$;

$(-A)(x)=A(-x)$.

Theorem 2.3 If L is a complete distributive lattice, then the intersection $\{A^k\}_{k\in I}$ of fuzzy algebra over fuzzy field F of Y is fuzzy algebra over fuzzy field F of Y; when I is a finitely set, $\forall$ $\alpha\in L$, $(\bigcap_{k\in I}A^k)_\alpha=\bigcap_{k\in I}A^k{}_\alpha$.

Proof: $\forall$ x, $y\in Y,\lambda\in X$.

$$(\bigcap_{k\in I}A^k)(x-y)=\bigwedge_{k\in I}A^k(x-y)\geq\bigwedge_{k\in I}(A^k(x)\wedge A^k(y))$$
$$=(\bigwedge_{k\in I}A^k(x))\wedge(\bigwedge_{k\in I}A^k(y))$$
$$=(\bigcap_{k\in I}A^k(x))\wedge(\bigcap_{k\in I}A^k(y))$$

$$(\bigcap_{k\in I}A^k)(x\cdot y)=\bigwedge_{k\in I}A^k(x\cdot y)\geq\bigwedge_{k\in I}(A^k(x)\wedge A^k(y))$$

$$=(\bigwedge_{k\in I} A^k(x))\wedge(\bigwedge_{k\in I} A^k(y))$$

$$=(\bigcap_{k\in I} A^k(x))\wedge(\bigcap_{k\in I} A^k(y))$$

$$(\bigcap_{k\in I} A^k)(\lambda x)=\bigwedge_{k\in I} A^k(\lambda x)\geq\bigwedge_{k\in I}(F(\lambda)\wedge A^k(x))$$

$$=F(\lambda)\wedge(\bigwedge_{k\in I} A^k(x))$$

$$=F(\lambda)\wedge(\bigcap_{k\in I} A^k(x))$$

$$(\bigcap_{k\in I} A^k)(0)=\bigwedge_{k\in I} A^k(0)=1$$

So $\bigcap_{k\in I} A^k$ is fuzzy algebra over fuzzy field F of Y.

Secondly, when I is a finitely set,$\forall\ \alpha\in L, x\in\bigcap_{k\in I} A^k{}_\alpha$, then $\forall\ k\in I, A^k{}_\alpha(x)\geq\alpha$, that is $\bigcap_{k\in I} A^k(x)\geq\alpha$, $x\in(\bigcap_{k\in I} A^k)_\alpha$, due to x's wantonness, we obtain $\bigcap_{k\in I} A^k{}_\alpha\subseteq(\bigcap_{k\in I} A^k)_\alpha$.

Conversely, $x\in(\bigcap_{k\in I} A^k)_\alpha$,that is $(\bigcap_{k\in I} A^k)_\alpha\geq\alpha$, so $\bigcap_{k\in I} A^k(x)\geq\alpha$, then $\forall\ k\in I$, $A^k(x)\geq\alpha$, so $\forall k\in I$, $x\in A^k{}_\alpha$, due to x's wantonness, we obtain $(\bigcap_{k\in I} A^k)_\alpha\subseteq\bigcap_{k\in I} A^k{}_\alpha$.

So $(\bigcap_{k\in I} A^k)_\alpha=\bigcap_{k\in I} A^k{}_\alpha$.

Definition 2.4. Algebra Y's fuzzy subset A is fuzzy algebra over fuzzy field F iif

(i) $\bigvee_{x\in Y} A(x)=1$;

(ii) $A+A\subseteq A, A\cdot A\subseteq A, \forall\lambda\in X, \lambda A\subseteq A, -A\subseteq A$.

Proof. Necessity: If A is fuzzy algebra subset over field F of Y, then $A(0)=1$, so $\bigvee_{x\in Y} A(x)=1$,that is (i) holds.

Further $\forall\ x\in Y$, $(A+A)(x)=\bigcup_{x_1+x_2=x}[A(x_1)\wedge A(x_2)]$, because A is fuzzy algebra over field F of Y, so

$A(x_1+x_2)\geq A(x_1)\wedge A(x_2)$,

$(A+A)(x)=\bigcup_{x_1+x_2=x}[A(x_1)\wedge A(x_2)]\leq\bigcup_{x_1+x_2=x} A(x_1+x_2)$,

then $A+A\subseteq A$.

Seminally $A\cdot A\subseteq A$.

Also $\forall\lambda\in X$, $x\in Y$,$\lambda A(x)=\bigcup_{\lambda x_1=x}[F(\lambda)\wedge A(x_1)]$, because A is fuzzy algebra over field F of Y, so $A(\lambda x)\geq F(\lambda)\wedge A(x)$, furthermore we have

$\lambda A(x)=\bigcup_{\lambda x_1=x}[F(\lambda)\wedge A(x_1)]\leq\bigcup_{\lambda x_1=x} A(\lambda x_1)=A(x)$, that is $\lambda A\subseteq A$.

Similarly $-A\subseteq A$.

Sufficiency: $\forall x,y\in Y,\lambda\in X$, because $A+A\subseteq A$, $A\cdot A\subseteq A, \forall\lambda\in X, \lambda A\subseteq A$, $-A\subseteq A$, we have

$A(x+y) \geq (A+A)(x+y) = \vee_{x_1+y_1=x+y} (A(x_1) \wedge A(y_1)) \geq A(x) \wedge A(y)$;

$A(x \cdot y) \geq (A \cdot A)(x+y) = \vee_{x_1 \cdot y_1 = x \cdot y} (A(x_1) \wedge A(y_1)) \geq A(x) \wedge A(y)$;

$A(\lambda x) \geq (\lambda A)(\lambda x) = \vee_{\lambda x_1 = \lambda x} (F(\lambda) \wedge A(x_1)) \geq F(\lambda) \wedge (\vee_{\lambda x_1 = \lambda x} A(x_1))$

$= F(\lambda) \wedge A(x)$;

$A(-x) \geq (-A)(-x) = A(x)$.

Also because $\forall x \in Y$, $0 = x + (-x)$, we have $A(0) = A(x+(-x)) \geq A(x) \wedge A(-x)$, so $A(0) \geq \vee_{x \in Y} A(x) = 1$, that is $A(0)=1$.

We can conclude A is fuzzy algebra over field F of Y.

Due to theorem 2.4, we easily prove the following theorem.

Theorem 2.5. If A and B are fuzzy algebras over field F of Y, then $A+B$ is also fuzzy algebra over field F of Y.

3 REFERENCES

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