GENERALIZATION OF THE LAW OF LARGE NUMBERS FOR FUZZY NUMBERS

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This paper presents a new result about the law of large numbers for fuzzy numbers in the framework of the possibility theory. It's a generalized version of this law, results are presented in terms of the additive generator of a triangular norm.
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Introduction. In the presented paper the generalized version of Fuller's law of larde numbers [2] is discussed. He showed that for the sequense X_1, $X_2, \ldots$ of fuzzy numbers of a symmetric triangular form with a common width d the law of large numbers works. For the definition of T-sum, Fuller applyed t-norm T, which is weaker than Hamacher's operator. In [3,4] Dombi's operator is used and the fuzzy law of large numbers for more general environment is shown as well.

Here t-norm representation theorem of Ling is used as a basic tool and the results are presented in terms of the additive generator of a triangular norm.

Note, that membership function a triangular fuzzy number $X = (m,d)$, is defined as $\mu(x) = 1-|x-m|/d$, if $m - d \leqslant x \leqslant m + d$; otherwise $\mu(x) = 0$ and d - is its width; m - is its modal value ($d>0$, $-\infty < m < \infty$).

Now, the grade of the possibilty of the statement: " [a,b] contains the value of X" is defined as [2]

$\mathrm{Pos}(a \ll X \ll b) = \sup_{a<x<b} \mu(x)$; And necessity is defined as

$\mathrm{Nes}(a \ll X \ll b) = 1- \mathrm{Pos}(X<a,\ X>b)$.

Function T: $[0;1] \times [0;1] \to [0;1]$ is t-norm, if T is commutative, associative, non-decreasing and $T(0,1) = 0$, $T(1,1) = 1$.

A t-norm will be called Archimedian if T is continuous and $T(u,u) < u$; $0 < u < 1$.

Examples of t-norm are Hamacher's and Dombi's operators [1]

$$T_H(u,v)=\frac{uv}{r+(1-r)(u+v-uv)},\quad r>0 \qquad T_D(u,v)=\left\{1+\left[\left(\frac{1-u}{u}\right)^p+\left(\frac{1-v}{v}\right)^p\right]^{1/p}\right\}^{-1},\quad p>0$$

T-sum of two fuzzy numbers is denoted as $S_T = (X_1+X_2)_T$ and its membership function is defined as

$$\mu_{S_T}(z) = \sup_{x+y=z} T(\mu_1(x),\ \mu_2(y))$$

Results.

Theorem 1. If T is Archimedian t-norm, $X_i = (m_i,d)$, then for any $\varepsilon>0$

$$\mathrm{Nes}\left[M_n-\varepsilon < \left(\frac{X_1+X_2+\ldots+X_n}{n}\right)_T < M_n+\varepsilon\right] = 1-f^{-1}\left(\min\ (f(0),\ n\cdot f(\mu_{S_n}(M_n+\varepsilon))\right)$$

f - is an additive generator of a triangular norm T,

f^{-1} - is its inverse.

$\mu_{S_n}(z)$ - is membership functin of the T-sum $S_n = \left(\frac{X_1+X_2+\ldots+X_n}{n}\right)_T$

$$M_n = \frac{m_1+m_2+\ldots+m_n}{n}.$$

As it's well know, if $\lim\limits_{n\to\infty} \mathrm{Nes}\ (M_n-\varepsilon < S_n < M_n+\varepsilon) = 1$, then the law of large numbers for fuzzy numbers works [2].

Proof this theorem is based on the following propositions.

Proposition 1. If T is Archimedian t-norm, $X_i = (m_i,d)$, then membership function T-sum $S_T = (X_1+X_2)_T$ is

$$\mu_{S_T}(z) = \sup_{x+y=z} T(\mu_1(x),\ \mu_2(y)) = f^{-1}\left(\min\ (f(0),\ 2\cdot f(\mu(z))\right)$$

$$\mu(z) = \max(0,\ 1-|z-(m_1+m_2)|/2d).$$

Proof of the proposition 1. Out of the definition of T-sum and Ling's theorem [1], for a fixed $z=z^*$ we have:

$$\mu_{S_T}(z^*) = \sup_{x+y=z} T(\mu_1(x),\ \mu_2(y)) = \sup_{x} T(\mu_1(x),\ \mu_2(z^*-x)) =$$

$$= \sup_{x}\ f^{-1}\Big(\min\ (f(0),\ f(\mu_1(x)) + f(\mu_2(z^*-x))\Big)$$

Let m1<m2. We'll consider the proof taking into consideration only the left parts of membership functions for fuzzy numbers X_1, X_2, which have the following type:

$$\mu_1(x) = \max(0,\ 1 + (x-m_1)/d),\ m_1-d \leqslant x \leqslant m_1;$$

$$\mu_2(y) = \max(0,\ 1 + (y-m_2)/d),\ m_2-d \leqslant y \leqslant m_2;$$

$$\mu_2(z^*-x)) = \max(0,\ 1 + (z^*-x-m_2)/d),\ z^*-m_2 \leqslant x \leqslant z^*-m_2+d;$$

Taking into account that an additive generator $f:X\text{-->}[0;1]$ is a continious and decreasing function with $f(1) = 0$, it's easy to see that $f(\mu_1(x))$ is a deacreasing, while $f(\mu_2(z^*-x))$ is an increasing function on the interval

$$\max(z^*-m_2;\ \ m_1-d) \leqslant x \leqslant \min(z^*-m_2+d;\ m_1).$$

Then the sum: $f(\mu_1(x)) + f(\mu_2(z^*-x))$ will have a minimum value equal to

$$2\cdot f\Big(\max(0,\ 1 + (z^* - (m_1+m_2))/2d)\Big) = 2\cdot f\Big(\ \mu(z^*)\ \Big),$$

The minimum value can be reached by x, which can be found as a solution of the following equation: $1+(x-m_1)/d = 1+(z^*-x-m_2)/d$; $x=(z^*+m_1-m_2)/2$.

The same holds true for the right parts of membership functions.

Now it is clear that

$$\mu_{S_T}(z) = f^{-1}\Big(\min\ (f(0),\ 2\cdot f(\mu(z))\Big)$$

Proposition 1 is proved.

Proposition 2. If T is Archimedian t-norm, $X_i = (m_i, d)$, then membership function of T-sum S_n is

$$\mu_{S_n}(z) = f^{-1}\Big(\min\,(f(0),\ n\cdot f(\mu_n(z)))\Big),\ \mu_n(z) = \max(0,\ 1-|z-M_n|/d)$$

Proof of the proposition 2 is based on the results of the proposition 1.

Now out of the propositions 1,2 and using inequality from [2],[4] we can get the following

$$\mathrm{Nes}(|S_n-M_n| \ll \varepsilon) = 1 - \mathrm{Pos}(|S_n-M_n| > \varepsilon) = 1 - \sup_{\substack{z \\ |z-M_n|>\varepsilon}} \mu_{S_n}(z) =$$

$$= 1 - \sup_{\substack{z \\ |z-M_n|>\varepsilon}} f^{-1}\Big(\min\,(f(0),\ n\cdot f(\max(0,\ 1-|z-M_n|/d))\Big) =$$

$$= 1-f^{-1}\Big(\min\,(f(0),\ n\cdot f(\mu_{S_n}(M_n+\varepsilon))\Big) = 1-f^{-1}\Big(\min\,(f(0),\ n\cdot f(1-\varepsilon/d)\Big)$$

Which completes the proof of the theorem 1.

Examples. We'll consider some examples using our theorem.

1. As a triangular norm T we'll choose Yager's operator [1]:

$$T_Y(u,v) = 1 - \min\Big(1,\ (1-u)^q + (1-v)^q\Big)^{1/q}, \quad 0 \leqslant q < \infty$$

its additive generator is $f(x) = (1-x)^q$, $f(0)=1$, $f^{-1}(y)=1-y^{1/q}$.

Using that we'll calculate the right part of our theorem 1

$$f\Big(\mu_{S_n}(M_n+\varepsilon)\Big) = \Big(1 - (1-\varepsilon/d)\Big)^q = \Big(\varepsilon/d\Big)^q,\quad f^{-1}\Big(\min\,(f(0),\ n\cdot(\varepsilon/d)^q\Big) =$$

$$= 1 - \Big(\min\,(1,\ n\cdot\varepsilon/d)^q\Big)^{1/q} = \max\,(0,\ 1 - n^{1/q}\cdot(\varepsilon/d)).$$

Hence, the law of large numbers in this case works.

If we will consider a special case when $q = \infty$ then we will have Nes ($M_n-\xi \leqslant S_n \leqslant M_n+\xi$) = ξ/d. Hence the fuzzy law of large numbers does not work (see [2] - [4]).

2. As a triangular norm T we'll choose Hamaher's operator with r = 0. Its additive generator is $f(x) = (1-x)/x$, $f(0) = \infty$

$f^{-1}(y) = 1/(y+1)$. That's why $n\cdot f(1-\xi/d) = (n\cdot\xi/d) / (1-\xi/d)$,

$$f^{-1}\Big((n\cdot\xi/d)/(1-\xi/d)\Big) = \frac{1-\xi/d}{1 + (n-1)\xi/d}$$

Therefore $\lim_{n\to\infty}$ Nes ($M_n-\xi \leqslant S_n \leqslant M_n+\xi$) = 1 and the fuzzy law of large numbers works (see also [2] - [4]).

Note. In this paper we have considered the case when fuzzy numbers have symmetric triangular membership function. Out of the proof of the theorem 1, proposition 1 and 2 and out of the considered examples as well, we can note that the theorem 1 is also true for membership functions that satisfy the following conditions:

1. $\mu(x)$ has a modal value m, $\mu(m)=1$;
2. $\mu(x)$ is symmetric around m, $\mu(x-m) = \mu(m-x)$;
3. $\mu(x)$ is an increasing on the interval [m-d; m], $-\infty \leqslant m-d$.

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