Monotonic fuzzy implications *

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1 Introduction

Many valued logic initiated by Łukasiewicz [11] uses many-valued connectives from $[0,1]^2$ to $[0,1]$. only truth-functional These connectives are used in fuzzy set theory as a base of fuzzy logic (cf. Baldwin, Pilsworth [1]). In particular diverse generalizations of implication are represented by fuzzy relations in $[0,1]$. Recently one can find a long list of formulas representing fuzzy implications (cf. Kiszka, Kochańska, Śliwińska [10] or Cordón, Herrera, Peregrin [3]). Simultaneously there are published many set of axioms describing necessary properties of fuzzy implications (c.f. Baldwin, Pilsworth [1], Dubois, Prade [6] or Fodor, Roubens [7]). We use here the simplest set of axioms presented by Fodor and Roubens [7].

Definition 1. Any function $I\colon [0,1]^2 \to [0,1]$ is called *fuzzy implication* if it fulfils the following conditions:

I1. $\forall_{x,y,z\in[0,1]}\ (x \leqslant z \Rightarrow I(x,y) \geqslant I(z,y))$,

I2. $\forall_{x,y,z\in[0,1]}\ (y \leqslant z \Rightarrow I(x,y) \leqslant I(x,z))$,

I3. $\forall_{y\in[0,1]}\ I(0,y) = 1$,

I4. $\forall_{x\in[0,1]}\ I(x,1) = 1$,

I5. $I(1,0) = 0$.

*Paper entered to volume Fuzzy Systems in Medicine (Eds. P.S.Szczepaniak, P.J.G.Lisboa, S.Tsumoto) in Studies in Fuzziness and Soft Computing.

Set of all fuzzy implications will be denoted by FI and set of all continuous fuzzy implications is denoted by CFI.

It is evident that a good generalization of the crisp implication must fulfil the binary implication truth table, i.e.

$$I(0,0) = I(0,1) = I(1,1) = 1, \quad I(1,0) = 0. \tag{1}$$

We see that axioms I3-I5 guarantee (1). Conversely, conditions (1) with axioms I1, I2 suffice for validity of axioms I3 - I5. Namely we have

Lemma 1. *Function $I\colon [0,1]^2 \to [0,1]$ fulfilling (1) is a fuzzy implication iff it is monotonic with respect to both variables.*

By virtue of this lemma we can use the name "monotonic implications" as the characteristic property of the family FI. Moreover, for verification of axioms I1-I5 it suffice to verify (1) and monotonicity of I.

Example 1. The most frequently used implication functions are usually listed with suitable author's name. We have put here six famous implication functions completed e.g. by Dubois, Prade [6]. All of them fulfil (1) and are monotonic in both variables, so they belong to FI.

1. Łukasiewicz implication ([11])

$$I_{\mathrm{LK}}(x,y) = \min(1-x+y,1) = \begin{cases} 1 & \text{, if } x \leqslant y \\ 1-x+y & \text{, if } x > y \end{cases}, \quad x,y \in [0,1]. \tag{2}$$

2. Reichenbach implication ([12])

$$I_{RC}(x,y) = 1 - x + xy, \qquad x,y \in [0,1]. \tag{3}$$

3. Gödel implication ([9])

$$I_{GD}(x,y) = \begin{cases} 1 & \text{, if } x \leqslant y \\ y & \text{, if } x > y \end{cases}, \qquad x,y \in [0,1]. \tag{4}$$

4. Dienes implication ([5])

$$I_{DN}(x,y) = \max(1-x, y), \qquad x,y \in [0,1]. \tag{5}$$

5. Goguen implication ([8])

$$I_{GG}(x,y) = \begin{cases} 1 & , \text{ if } x = 0 \\ \min(1, \frac{y}{x}) & , \text{ if } x > 0 \end{cases} = \begin{cases} 1 & , \text{ if } x \leqslant y \\ \frac{y}{x} & , \text{ if } x > y \end{cases}, \quad x,y \in [0,1]. \tag{6}$$

6. Rescher implication ([13])

$$I_{RS}(x,y) = \begin{cases} 1 & , \text{ if } x \leqslant y \\ 0 & , \text{ if } x > y \end{cases}, \qquad x,y \in [0,1]. \tag{7}$$

Our investigations were inspired by paper of Czogała, Łęski [4] were they ask for relative location of implications (2)-(7). Many formulas for fuzzy implications did not give elements of FI. For example use formulas (I7) and (I12) from [3]:

$$I(x,y) = \max(1-x, \min(x,y)) \qquad x,y \in [0,1],$$

$$I(x,y) = \max(0, y-x) \qquad x,y \in [0,1].$$

The first example fulfils (1) but is not monotonic with respect to x. The second example is monotonic but does not fulfil (1) (I(0,0)=I(1,1)=0).

2 Lattice of fuzzy implications

The lattice properties of fuzzy implications family are following.

Theorem 1. *Family* $(FI, \min, \max)$ *is a complete, completely distribiutive lattice.*

Corollary 1. *FI has the greatest element*

$$I_1(x,y) = \begin{cases} 1 & , \textit{ if } x < 1 \textit{ or } y > 0 \\ 0 & , \textit{ if } x = 1 \textit{ and } y = 0 \end{cases}, \qquad x,y \in [0,1], \tag{8}$$

and the least element

$$I_0(x,y) = \begin{cases} 1 & , \textit{ if } x = 0 \textit{ or } y = 1 \\ 0 & , \textit{ if } x > 0 \textit{ and } y < 1 \end{cases}, \qquad x,y \in [0,1]. \tag{9}$$

Theorem 2. *Family* $(CFI, \min, \max)$ *is a distributive lattice (a sublattice of* $(FI,$ $\min, \max)$*).*

However lattice CFI is not complete. It follows from known fact that sequences of continuous functionscan have limits which are not continuous (cf. also Example 4).

Theorem 3. *Fuzzy implications (2)-(7) form two following chains:*

$$I_{DN} \leqslant I_{RC} \leqslant I_{LK}, \tag{10}$$

$$I_{RS} \leqslant I_{GD} \leqslant I_{GG} \leqslant I_{LK}. \tag{11}$$

3 Convexity of fuzzy implications family

Definition 2. Subet X of linear space is *convex* over $\mathbb{R}$ if with any two points $x, y \in X$, X contains line segment between x and y i.e.

$$\forall_{\lambda \in [0,1]} \; z = \lambda x + (1 - \lambda) y \in X.$$

Theorem 4. FI *and* CFI *are convex sets of functions.*

The above theorem brings a tool for generation of parametrized families of fuzzy implications. E.g. the first segment in chain (11) can be parametrized by

$$I_\lambda = \lambda I_{GD} + (1 - \lambda) I_{RS}, \qquad I_\lambda(x, y) = \begin{cases} 1 & , \text{ if } x \leqslant y \\ \lambda y & , \text{ if } x > y \end{cases}, \quad x, y \in [0,1],$$

for $\lambda \in [0,1]$. In the same way we can consider multidimensional simplexes of fuzzy implications.

4 Contrapositive implications

Definition 3 ([7]). By *reciprocal* function of $I \in FI$ we call I',

$$I'(x, y) = I(1 - y, 1 - x) \qquad x, y \in [0,1]. \tag{12}$$

Implication I is called contrapositive if $I' = I$.

Theorem 5. *The reciprocal function of an implication $I \in FI$ is also an implication and the same holds for continuous implications. ($I \in CFI \implies I' \in CFI$)*

Example 2. Among six fuzzy implications from Example 1 and two from Corollary 1 we have six contrapositive examples: $I_0' = I_0$, $I_1' = I_1$, $I_{RS}' = I_{RS}$, $I_{\mathrm{LK}}' = I_{\mathrm{LK}}$, $I_{RC}' = I_{RC}$, $I_{DN}' = I_{DN}$, and we obtain two new implications

$$I'_{GD}(x,y) = \begin{cases} 1 & , \text{ if } x \leqslant y \\ 1-x & , \text{ if } x > y \end{cases}, \qquad x,y \in [0,1], \tag{13}$$

$$I'_{GG}(x,y) = \begin{cases} 1 & , \text{ if } y = 1 \\ \min(1, \frac{1-x}{1-y}) & , \text{ if } y < 1 \end{cases} = \begin{cases} 1 & , \text{ if } x \leqslant y \\ \frac{1-x}{1-y} & , \text{ if } x > y \end{cases}, \qquad x,y \in [0,1]. \tag{14}$$

Lemma 2. *Let $I, J \in FI$. Operation defined by (10) is order preserving (isotone), i.e.*

$$I \leqslant J \Rightarrow I' \leqslant J' \tag{15}$$

and for $I_t \in FI$, $t \in T$ we get

$$(\sup_{t\in T} I_t)' = \sup_{t\in T} I_t', \qquad (\inf_{t\in T} I_t)' = \inf_{t\in T} I_t'. \tag{16}$$

Theorem 6. *Set of all contrapositive fuzzy implications is a complete, completely distributive lattice and set of all continuous contrapositive fuzzy implications is a distributive lattice.*

Examples of contrapositive implications can be obtained not only as lattice sum or product of given contrapositive implications. Another way is a combination of reciprocal functions.

Lemma 3. *For any $I \in FI$ functions $\min(I, I')$, $\max(I, I')$ are contrapositive implications.*

Example 3. Using fuzzy implications (3), (5), (11) and (12) we obtain four contrapositive implications:

$$I_2 = I_{GG} \vee I'_{GG}, \qquad I_2(x,y) = \begin{cases} 1 & , \text{ if } x \leqslant y \\ \max(\frac{y}{x}, \frac{1-x}{1-y}) & , \text{ if } x > y \end{cases} \tag{17}$$

$$I_3 = I_{GG} \wedge I'_{GG}, \qquad I_3(x,y) = \begin{cases} 1 & \text{, if } x \leqslant y \\ \min(\frac{y}{x}, \frac{1-x}{1-y}) & \text{, if } x > y \end{cases} \tag{18}$$

$$I_4 = I_{GD} \vee I'_{GD}, \qquad I_4(x,y) = \begin{cases} 1 & \text{, if } x \leqslant y \\ \max(1-x, y) & \text{, if } x > y \end{cases} \tag{19}$$

$$I_5 = I_{GD} \wedge I'_{GD}, \qquad I_5(x,y) = \begin{cases} 1 & \text{, if } x \leqslant y \\ \min(1-x, y) & \text{, if } x > y \end{cases} \tag{20}$$

Another way of generating contrapositive implications is getting convex combinations of given examples of implications. Since formula (10) leads us to

$$(\lambda I + (1-\lambda)J)' = \lambda I' + (1-\lambda)J', \qquad \text{for } I, J \in FI, \lambda \in [0,1]$$

then we get

Theorem 7. *Set of all contrapositive fuzzy implications is convex.*

5 Selfconjugate implications

Definition 4. Let $\varphi\colon [0,1] \to [0,1]$ be an increasing bijection, $I \in FI$. We say that the function

$$I^*(x,y) = I^*_\varphi(x,y) = \varphi^{-1}(I(\varphi(x), \varphi(y))), \qquad x, y \in [0,1] \tag{21}$$

is *φ-conjugate* to I. Implication $I \in FI$ is called φ-selfconjugate if $I^*_\varphi = I$ and selfconjugate (absolutely) if $I^*_\varphi = I$ for all φ.

Theorem 8. *Let $\varphi\colon [0,1] \to [0,1]$ be an increasing bijection. For any $I \in FI$ ($I \in CFI$)*

$$I^*_\varphi \in FI \;\; (I^*_\varphi \in CFI). \tag{22}$$

Example 4. Let $\varphi\colon [0,1] \to [0,1]$ be an increasing bijection. For six implications from Example 1 and two from Corollary 1 we have: $I_0^* = I_0$, $I_1^* = I_1$, $I_{RC}^* = I_{RC}$, $I_{GD}^* = I_{GD}$. So this implications are selfconjugate. For next four implications we get new fuzzy implications

$$I^*_{GG}(x,y) = \begin{cases} 1 & , \text{ if } x \leqslant y \\ \varphi^{-1}(\frac{\varphi(y)}{\varphi(x)}) & , \text{ if } x > y \end{cases}, \tag{23}$$

$$I^*_{DN}(x,y) = \max(\varphi^{-1}(1-\varphi(x)), y), \tag{24}$$

$$I^*_{\mathrm{LK}}(x,y) = \min(\varphi^{-1}(1-\varphi(x)+\varphi(y)), 1), \tag{25}$$

$$I^*_{RC}(x,y) = \varphi^{-1}(1-\varphi(x)+\varphi(x)\varphi(y)). \tag{26}$$

Now we can give examples of sequences of continuous implications which limits are not contionuous. Let $\varphi(x) = x^n$, $n \in \mathbb{N}$. We get

$$I^1_n(x,y) = I^*_{\mathrm{LK},n}(x,y) = \min(1, \sqrt[n]{1-x^n+y^n}), \qquad n \in \mathbb{N},\ x,y \in [0,1],$$

$$I^2_n(x,y) = I^*_{RC,n}(x,y) = \sqrt[n]{1-x^n+x^ny^n}, \qquad n \in \mathbb{N},\ x,y \in [0,1],$$

$$I^3_n(x,y) = I^*_{DN,n}(x,y) = \max(\sqrt[n]{1+x^n}, y), \qquad n \in \mathbb{N},\ x,y \in [0,1].$$

These sequences are convergent and

$$\lim_{n\to\infty} I^1_n(x,y) = I_1(x,y),$$

$$\lim_{n\to\infty} I^2_n(x,y) = \lim_{n\to\infty} I^3_n(x,y) = \begin{cases} 1 & , \text{ if } x < 1 \\ y & , \text{ if } x = 1 \end{cases}.$$

Lemma 4. *Let $I, J \in FI$. Operation defined by (20) is order preserving (isotone), i.e.*

$$I \leqslant J \Leftrightarrow I' \leqslant J' \tag{27}$$

and for $I_t \in FI$, $t \in T$ we get

$$(\sup_{t\in T} I_t)^* = \sup_{t\in T} I^*_t, \qquad (\inf_{t\in T} I_t)^* = \inf_{t\in T} I^*_t. \tag{28}$$

Theorem 9. *Set of all selfconjugate fuzzy implications is a complete, completely distributive lattice, and set of all continuous selfconjugate implications is a distributive lattice.*

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