

Some Results about F —homomorphism of F_R^A —modules

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Abstract:

In this paper, we prove the first isomorphism theorem, the second isomorphism theorem, the extension theorem of isomorphism, and another some important properties of F_R^A -module.

Keywords:

F_R^A -module, complete lattice, F —homomorphism, F —isomorphism.

1. Note

The concept of F_R^A -modules was introduced by Zhao Jianli [2]. The paper [1], [2], [3], [4] gives the important properties of F_R^A -modules and F_R^A -module categories, and establishes the basic theory of F_R^A -modules. This will promote study of fuzzy homological algebra. In this paper, we will carry on the work of [1], [2], [3], [4], we will prove the first isomorphism theorem, the second isomorphism theorem, the extension theorem of isomorphism, and another some important properties of F_R^A -module.

Let X be any nonempty set, and L a complete distributive lattice with 0 and 1. A fuzzy subset A on X is characterized by a mapping $A: X \rightarrow L$, X^L denotes the set of whole fuzzy subset of X . In this paper, R is a ring with identity $1 \neq 0$ and each module which involved is an unitary R -module.

Definition 1.1 [2] Let R be a ring and M a left R -module. Let A be a fuzzy ring of R , $B_M \in M^L$, then B_M is called a fuzzy submodules over fuzzy subring A , if

$$(1) B_M(x+y) \geq B_M(x) \wedge B_M(y),$$

$$(2) B_M(-x) \geq B_M(x)$$

$$(3) B_M(0) = 1$$

$$(4) B_M(rx) \geq A(r) \wedge B_M(x)$$

for all $x, y \in M, r \in R$. In brief B_M is an F_R^A -module of M (or F_R^A -module).

Definition 1.2 [2] Let N be an R -submodule of M , B_M and C_N be an F_R^A -module of M and N respectively, if for all $x \in N$, $B_M(x) \geq C_N(x)$, then C_N is called F_R^A -submodule of B_M , write by $C_N \leq B_M$.

2. Some Results about F -homomorphisms of F_R^A -modules

Theorem 2.1 [First isomorphism theorem] Let $\tilde{f}: B_M \rightarrow C_N$ be an F -homomorphism. Then there is a unique F -isomorphism $\tilde{\eta}: B_M/\ker \tilde{f} \rightarrow \text{Im } \tilde{f}$ such that the diagram 1

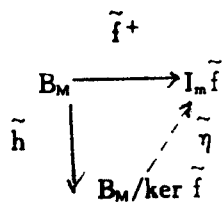


diagram 1

is commutative, where h is natural R -homomorphisms from M into $M/\ker f$, f^+ denote the R -homomorphisms from M into $\text{Im } f$ and $f(x) = f^+(x)$, for all $x \in M$.

Proof. Applying Corollary 4.9 of [1] in the case where $C_N = \text{Im } \tilde{f}$, $C_N^1 = \bar{0}$ and $B_M^1 = \ker \tilde{f}$, we obtain the existence of a unique F -homomorphisms $\tilde{\eta}: B_M / \ker \tilde{f} \rightarrow \text{Im } \tilde{f}$ such that $\tilde{\eta}\tilde{h} = \tilde{f}^+$. Because \tilde{f}^+ is F -epimorphism, so also is $\tilde{\eta}$. Since $\ker f = f^-(0) = f^-(N_1)$, by corollary 4.9 of [1], $\tilde{\eta}$ is also F -homomorphism. Thus $\tilde{\eta}$ is an F -bimorphism. Moreover, for $\forall y \in \text{Im } f$

$$\begin{aligned}
 \tilde{\eta}(B_M / \ker \tilde{f})(y) &= \bigvee \{ (B_M / \ker \tilde{f})(\bar{x}) \mid \bar{x} \in M / \ker f, \eta(\bar{x}) = y \} \\
 &= \bigvee \{ B_M(v) \mid v \in \bar{x}, \bar{x} \in M / \ker f, \eta(\bar{x}) = y \} \\
 &= \text{Im } \tilde{f}(y)
 \end{aligned}$$

Consequently $\tilde{\eta}$ is an F -isomorphism.

Theorem 2.2 [Second isomorphism theorem] If B_M is an F_R^A -module and $C_N \leq B_M$, $D_P \leq C_N$, then $C_N / D_P \leq B_M / D_P$ and there is a unique F -isomorphism $\tilde{h}: B_M / C_N \rightarrow (B_M / D_P) / (C_N / D_P)$ such that the following diagram2 is commutative:

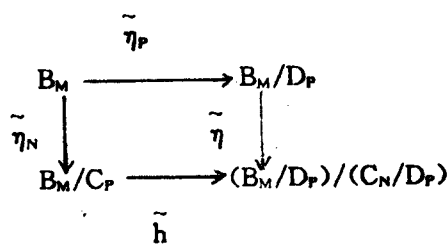


diagram 2

Proof. By Proposition 3.6 of [1], C_N / D_P , B_M / D_P is F_R^A -module of N/P and M/P respectively. For $\forall \bar{x} \in N/P$,

$$\begin{aligned}
 (C_N / D_P)(\bar{x}) &= \bigvee \{ C_N(v) \mid v \in \bar{x} \} \\
 &\leq \{ B_M(v) \mid v \in \bar{x} \} \\
 &\leq (B_M / D_P)(\bar{x})
 \end{aligned}$$

So $C_N / D_P \leq B_M / D_P$.

Applying the condition of theorem and second isomorphism theorem of R -module, there is a

unique R -isomorphism $h: M/N \rightarrow (M/P)/(N/P)$, such that the diagram 3

$$\begin{array}{ccc} M & \xrightarrow{\eta_P} & M/P \\ \eta_N \downarrow & & \downarrow \eta \\ M/N & \xrightarrow{h} & (M/P)/(N/P) \end{array}$$

diagram 3

is commutative.

Moreover, for all $x \in M$, let $\bar{x} = x + N$, $\bar{x}' = \bar{x} + P$, $\bar{\bar{x}} = \bar{x} + N/P$, then

$$\begin{aligned} \tilde{h}(B_M/C_N)(\bar{x}) &= (B_M/C_N)(\bar{x}) \\ &= \bigvee \{B_M(v) \mid v \in x + N\} \\ &= \bigvee \{B_M(v) \mid v \in u + P, \bar{u} \in \bar{x}' + N/P\} \\ &= \bigvee \{(B_M/D_P)(\bar{u}) \mid \bar{u} \in \bar{x}' + N/P\} \\ &= ((B_M/D_P)/(C_N/D_P))(\bar{\bar{x}}) \end{aligned}$$

thus \tilde{h} is an F -isomorphism. Since diagram 1 is commutative, so diagram 2 is commutative.

Theorem 2.3 Let $f_i: M \rightarrow N_i$ be an R -homomorphism and $B_{N_i}^i$ an F_R^\wedge -module, $i \in I$, then there is an F_R^\wedge -module B_M of M such that $\tilde{f}_i: B_M \rightarrow B_{N_i}^i$ is F -homomorphism, for all $i \in I$.

Proof. Let $B_M = \bigwedge_{i \in I} \tilde{g}_i^{-1}(B_{N_i}^i)$, then $\tilde{f}_i: B_M \rightarrow B_{N_i}^i$ is F -homomorphism, for all $i \in I$.

Theorem 2.4. Let $f_i: M_i \rightarrow N$ be an R -homomorphism, $B_{M_i}^i$ be an F_R^\wedge -module, $i \in I$. If $\sum_{i \in I} f_i(M_i) = \bigoplus_{i \in I} f_i(M_i) = K$, then there is an F_R^\wedge -module B_K of K such that $\tilde{f}_i: B_{M_i}^i \rightarrow B_K$ is an F -homomorphism, for all $i \in I$.

Proof. If $x = (\dots x'_i, \dots, x'_j, \dots) \in K$, $x'_i \in f_i(M_i)$, let $B_K(x) = \bigwedge_{i \in I} (f_i(B_{M_i}^i)(x'_i))$.

For all $x' = (\dots, x'_i, \dots, x'_j, \dots) \in K$, $y' = (\dots, y'_i, \dots, y'_j, \dots) \in K$.

where $x'_i, y'_i \in f_i(M_i)$, $i \in I$, $r \in R$, then

$$\begin{aligned} B_K(x' - y') &= \bigwedge_{i \in I} (\tilde{f}_i(B_{M_i}^i)(x'_i - y'_i)) \\ &\geq \bigwedge_{i \in I} (\tilde{f}_i(B_{M_i}^i)(x'_i)) \bigwedge_{i \in I} (\tilde{f}_i(B_{M_i}^i)(y'_i)) \\ &\geq B_K(x') \wedge B_K(y') \\ B_K(rx') &= \bigwedge_{i \in I} (\tilde{f}_i(B_{M_i}^i)(rx'_i)) \\ &\geq \bigwedge_{i \in I} (A(r) \wedge (\tilde{f}_i(B_{M_i}^i)(x'_i))) \\ &= A(r) \bigwedge_{i \in I} (\tilde{f}_i(B_{M_i}^i)(x'_i)) \\ &= A(r) \wedge B_K(x') \end{aligned}$$

thus B_K is an F_R^\wedge -module of K .

Moreover, for all $x'_i \in M_i, i \in I$,

$$\begin{aligned}\tilde{f}_i^{-1}(B_K)(x'_i) &= B_K(\cdots, 0, f_i(x_i), 0, \cdots) \\ &= \tilde{f}_i(B_K)(f_i(x_i)) \\ &\geq B_{M_i}(x_i)\end{aligned}$$

So $\tilde{f}_i: B_{M_i} \rightarrow B_K$ is an F-homomorphism.

Theorem 2.5 (Extension theorem of isomorphism) Let $f: M \rightarrow N$ be an R-isomorphism, B_M is an F_R^A -module of M , then there is F_R^A -module C_N of N such that $B_M \cong C_N$.

Proof. For all $x' \in N$, since f is an R-isomorphism, then there is a unique $x \in M$ such that $f(x) = x'$. Let $C_N: N \rightarrow L, x' \mapsto B_M(x)$, where $f(x) = x'$, then C_N be an F_R^A -module of N . In fact, for all $x', y' \in N$, there are $x, y \in M$ such that $f(x) = x', f(y) = y'$, then $f(x-y) = x'-y', f(rx) = rx'$. So

$$C_N(x'-y') = B_M(x-y) \geq B_M(x) \wedge B_M(y) = C_N(x') \wedge C_N(y')$$

$$C_N(0') = B_M(0) = 1$$

$$C_N(rx') = B_M(rx) \geq A(r) \wedge B_M(x) = A(r) \wedge C_N(x')$$

thus C_N be an F_R^A -module of N .

Applying definition of C_N , we have

$$C_N(f(x)) = C_N(x') = B_M(x)$$

Consequently, $B_M \cong C_N$.

Theorem 2.6. Let B_M be an F_R^A -module, then there is an F_R^A -module C_H of $H = \text{Hom}_R(R, M)$ such that $B_M \cong C_H$.

Proof. Let $\rho: M \rightarrow \text{Hom}_R(R, M), \rho(x)r = rx, r \in R$, then ρ is an R-isomorphism. Let $A' = A$, for all $f \in \text{Hom}_R(R, M)$, then there is a unique $x \in M$, such that $\rho(x) = f$, let $C_H(f) = B_M(x)$. By Theorem 2.5, $\tilde{\rho}: B_M \rightarrow C_H$ is an F-isomorphism, so $B_M \cong C_H$.

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