Some Results about F—homomorphism of F_R^A—modules

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Abstract:

In this paper, we prove the first isomorphism theorem, the second isomorphism theorem, the extension theorem of isomorphism, and another some important properties of F_R^A-module.

Keywords:

F_R^A-module, complete lattice, F—homomorphism, F—isomorphism.

1. Note

The concept of F_R^A-modules was introduced by Zhao Jianli [2]. The paper [1], [2], [3], [4] gives the important properties of F_R^A-modules and F_R^A-module categories, and establishe the basic theory of F_R^A-modules. This will promote study of fuzzy homological algebra. In this paper, we will carry on the work of [1], [2], [3], [4], we will prove the fivst isomorphism theorem, the second isomorphism theorem, the extension theorem of isomorphism, and another some important propertise of F_R^A-module.

Let X be any nonempty set, and L a complete distributive lattice with 0 and 1. A fuzzy subset A on X is characterized by a mapping $A: X \to L$, X^L denotes the set of whole fuzzy subset of X. In this paper, R is a ring with identity $1 \neq 0$ and each module which involoed is an unitary R-module.

Definition1.1 [2] Let R be a ring and M a left R-module. Let A be a fuzzy ring of R, $B_M \in M^L$, then B_M is called a fuzzy submodules over fuzzy subring A, if

(1) $B_M(x+y) \geqslant B_M(x) \wedge B_M(y)$,

(2) $B_M(-x) \geqslant B_M(x)$

(3) $B_M(0) = 1$

(4) $B_M(rx) \geqslant A(r) \wedge B_M(x)$

for all $x, y \in M$, $r \in R$, In brief B_M is an F_R^A-module of M(or F_R^A-module).

Definition 1.2[2] Let N be an R-submodule of M, B_M and C_N be an F_R^A-module of M and N respectively, if for all $x \in N$, $B_M(x) \geqslant C_N(x)$, then C_N is called F_R^A-submodule of B_M, write by $C_N \leqslant B_M$.

2. Some Results about F-homomorphisms of F_R^A-modules

Theorem 2.1 [First isomorphism theorem] Let $\tilde{f}: B_M \longrightarrow C_N$ be an F-homomorphism. Then there is a unique F-isomorphism $\tilde{\eta}: B_M/\ker \tilde{f} \longrightarrow \operatorname{Im} \tilde{f}$ such that the diagram 1

$$\begin{tikzcd} B_M \arrow[r, "\tilde{f}^+"] \arrow[d, "\tilde{h}"'] & I_m\tilde{f} \\ B_M/\ker\tilde{f} \arrow[ur, dashed, "\tilde{\eta}"'] & \end{tikzcd}$$

diagram 1

is commutative, where h is natural R-homomorphisms from M into M/kerf, f^+ denote the R-homomorphisms from M into Imf and $f(x)=f^+(x)$, for all $x\in M$.

Proof. Applying Corollary 4.9 of [1] in the case where $C_N=\mathrm{Im}\ \tilde{f}$, $C^1_{N_1}=\tilde{0}$ and $B^1_{M_1}=\ker\tilde{f}$, we obtain the existence of a unique F-homomorphisms $\tilde{\eta}: B_M/\ker\tilde{f}\to\mathrm{Im}\ \tilde{f}$ such that $\tilde{\eta}\tilde{h}=\tilde{f}^+$. Because $\tilde{f}^+$ is F-epimorphism, so also is $\tilde{\eta}$. Since $\ker f=f^{\leftarrow}(0)=f^{\leftarrow}(N_1)$, by corollary 4.9 of [1], $\tilde{\eta}$ is also F-momomophism. Thus $\tilde{\eta}$ is an F-bimorphism. Moreover, for $\forall\ y\in\mathrm{Imf}$

$$\begin{aligned} \tilde{\eta}\,(B_M/ker\tilde{f})(y) &= \vee\ \{(B_M/ker\tilde{f})(\bar{x})\,|\,\bar{x}\in M/kerf,\eta(\bar{x})=y\} \\ &= \vee\ \{B_M(v)\,|\,v\in\bar{x},\bar{x}\in M/kerf,\eta(\bar{x})=y\} \\ &= \mathrm{Im}\tilde{f}\,(y) \end{aligned}$$

Consequently $\tilde{\eta}$ is an F-isomorphism.

Theorem 2.2 [Second isomorphism theorem] If B_M is an F^A_R-module and $C_N\leqslant B_M$, $D_P\leqslant C_N$, then $C_N/D_P\leqslant B_M/D_P$ and there is a unique F-isomorphism $\tilde{h}: B_M/C_N\longrightarrow(B_M/D_P)/(C_N/D_P)$ such that the following diagram2 is commutative:

$$\begin{tikzcd} B_M \arrow[r, "\tilde{\eta}_P"] \arrow[d, "\tilde{\eta}_N"'] & B_M/D_P \arrow[d, "\tilde{\eta}"'] \\ B_M/C_P \arrow[r, "\tilde{h}"'] & (B_M/D_P)/(C_N/D_P) \end{tikzcd}$$

diagram 2

Proof. By Proposition 3.6 of [1], C_N/D_P, B_M/D_P is F^A_R-module of N/P and M/P respectively. For $\forall\ \bar{x}\in N/P$,

$$\begin{aligned} (C_N/D_P)(\bar{x}) &= \vee\ \{C_N(v)\,|\,v\in\bar{x}\} \\ &\leqslant\{B_M(v)\,|\,v\in\bar{x}\} \\ &\leqslant(B_M/D_P)(\bar{x}) \end{aligned}$$

So $C_N/D_P\leqslant B_M/D_P$.

Applying the condition of theorem and second isomorphism theorem of R-module, there is a unique R-isomorphism $h: M/N \longrightarrow (M/P)/(N/P)$, such that the diagram 3

$$\begin{tikzcd} M \arrow[r, "\eta_P"] \arrow[d, "\eta_N"'] & M/P \arrow[d, "\eta"'] \\ M/N \arrow[r, "h"'] & (M/P)/(N/P) \end{tikzcd}$$

diagram 3

is commutative.

Moreover, for all $x \in M$, let $\bar{x} = x + N$, $x' = x + P$, $\bar{x} = x + N/P$, then

$$\begin{aligned} \tilde{h}\,(B_M/C_N)(\bar{x}) &= (B_M/C_N)(\bar{x}) \\ &= \vee\ \{B_M(v) \mid v \in x + N\} \\ &= \vee\ \{B_M(v) \mid v \in u + P, \bar{u} \in \bar{x}' + N/P\} \\ &= \vee\ \{(B_M/D_P)(\bar{u}) \mid \bar{u} \in \bar{x}' + N/P\} \\ &= ((B_M/D_P)/(C_N/D_P))(\bar{x}) \end{aligned}$$

thus $\tilde{h}$ is an F-isomorphism. Since diagrams 1 is commutative, so diagram 2 is commutative.

Theorem 2.3 Let $f_i: M \longrightarrow N_i$ be an R-homomorphism and $B^i_{N_i}$ an F^A_R-module, $i \in I$, then there is an F^A_R-module B_M of M such that $\tilde{f}_i: B_M \longrightarrow B^i_{M_i}$ is F-homomorphism, for all $i \in I$.

Proof. Let $B_M = \bigwedge_{i \in I} \tilde{g}_i^{-1}(B^i_{N_i})$, then $\tilde{f}_i: B_M \longrightarrow B^i_{N_i}$ is F-homomorphism, for all $i \in I$.

Theorem2.4. Let $f_i: M_i \longrightarrow N$ be an R-homomorphism, $B^i_{M_i}$ be an F^A_R-module, $i \in I$. If $\sum_{i \in I} f_i(M_i) = \bigoplus_{i \in I} f_i(M_i) = K$, then there is an F^A_R-module B_K of K such that $\tilde{f}_i: B^i_{M_i} \longrightarrow B_K$ is an F-homomorphism, for all $i \in I$.

Proof. If $x = (\cdots x'_i, \cdots, x'_j, \cdots) \in K$, $x'_i \in f_i(M_i)$, let $B_K(x) = \bigwedge_{i \in I} (f_i(B^i_{M_i})(x'))$,

For all $x' = (\cdots, x'_i, \cdots, x'_j, \cdots) \in K$, $y' = (\cdots, y'_i, \cdots, y'_j, \cdots) \in K$.

where $x'_i, y'_i \in f_i(M_i)$, $i \in I$, $r \in R$, then

$$\begin{aligned} B_K(x'-y') &= \bigwedge_{i \in I} (\tilde{f}_i\,(B^i_{M_i})(x'_i - y'_i)) \\ &\geqslant \bigwedge_{i \in I} (\tilde{f}_i\,(B^i_{M_i})(x'_i)) \bigwedge_{i \in I} (\tilde{f}_i\,(B^i_{M_i})(y'_i)) \\ &\geqslant B_K(x') \wedge B_K(y') \\ B_K(rx') &= \bigwedge_{i \in I} (\tilde{f}_i\,(B^i_{M_i})(rx')) \\ &\geqslant \bigwedge_{i \in I} (A(r) \wedge (\tilde{f}_i\,(B^i_{M_i})(x')) \\ &= A(r) \bigwedge_{i \in I} (\tilde{f}_i\,(B^i_{M_i})(x')) \\ &= A(r) \wedge B_K(x') \end{aligned}$$

thus B_K is an F^A_R-module of K.

Moreover, for all $x'_i \in M_i, i \in I$,

$$\begin{aligned}\tilde{f}_i^{-1}(B_K)(x'_i) &= B_K(\cdots,0,f_i(x_i),0,\cdots)\\ &= \tilde{f}_i(B_K)(f_i(x_i))\\ &\geqslant B'_{M_i}(x_i)\end{aligned}$$

So $\tilde{f}_i : B'_{M_i} \longrightarrow B_K$ is an F-homomorphism.

Theorem 2.5 (Extension theorem of isomorphism) Let $f: M \longrightarrow N$ be an R-isomorphism, B_M is an F_R^A-module of M, then there is F_R^A-module C_N of N scuh that $B_M \cong C_N$.

Proof. For all $x' \in N$, since f is an R-isomorphism, then there is a unique $x \in N$ such that $f(x) = x'$. Let $C_N: N \longrightarrow L, x' \longrightarrow B_M(x)$, where $f(x) = x'$, then C_N be an F_R^A-module of N. In fact, for all x', $y' \in N'$, there are $x, y \in M$ such that $f(x) = x', f(y) = y'$, then $f(x-y) = x'-y', f(rx) = rx'$. So

$C_N(x'-y') = B_M(x-y) \geqslant B_M(x) \wedge B_M(y) = C_N(x') \wedge C_N(y')$

$C_N(0') = B_M(0) = 1$

$C_N(rx') = B_M(rx) \geqslant A(r) \wedge B_M(x) = A(r) \wedge C_N(x')$

thus C_N be an F_R^A-module of N.

Applying definition of C_N, we have

$$C_N(f(x)) = C_N(x') = B_M(x)$$

Consequently, $B_M \cong C_N$.

Theorem2.6. Let B_M be an F_R^A-module, then there is an F_R^A-module C_H of $H = Hom_R(R, M)$ such that $B_M \cong C_H$.

Proof. Let $\rho: M \longrightarrow Hom_R(R, M), \rho(x)r = rx, r \in k$, then ρ is an R-isomorphism. Let $A' = A$, for all $f \in Hom_R(R, M)$, then there is a unique $x \in M$, such that $\rho(x) = f$, let $C_H(f) = B_M(x)$. By Theorem 2.5, $\tilde{\rho}: B_M \longrightarrow C_H$ is an F-isomorphism, so $B_M \cong C_H$.

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