

On the homomorphism of the T_H - interval valued fuzzy subgroups^{*}

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Abstract In this paper, on the space of the interval valued fuzzy set on groups, We define the interval valued fuzzy subgroup by introducing the idempotent interval norm, and applying the extension principle of interval valued fuzzy sets given by [5], We discuss the problems of its homomorphic images and inverse images.

Keywords Interval valued fuzzy sets; idempotent interval norms; interval valued fuzzy extensive groups; interval valued fuzzy subgroups.

1 Basic concepts

An interval valued fuzzy set was presented by B. Gorzafczany [1] and B. Turksen [2] at first, In the sense of the minimal operator " \wedge ", R. Biswas [4] gave the definition of fuzzy valued fuzzy subgroups. In this paper, in the sense of operator T_H being more extensive than operator " \wedge ", We define the T_H -interval valued fuzzy subgroup, and discuss the problem of its homomorphic images.

Throughout this paper, let I be a closed unit interval, i. e. , $I = [0, 1]$.

Let $[I] = \{\bar{a} = [a^-, a^+] \mid a^- \leq a^+, a^-, a^+ \in I\}$,

On the operations and orders of the elements in $[I]$ are defined as folloves:

For arbitrary $\bar{a}_j \in [I]$, $\bar{a}_j = [a_j^-, a_j^+]$, $a_j^-, a_j^+ \in I$, $j \in J$.

$$\bigvee_{j \in J} \bar{a}_j = \sup\{a_j^- \mid j \in J\}; \quad \sup \bar{a}_j = [\bigvee_{j \in J} a_j^-, \bigvee_{j \in J} a_j^+].$$

Especially, for arbitrary $\bar{a}, \bar{b} \in [I]$, $\bar{a} = [a^-, a^+]$, $\bar{b} = [b^-, b^+]$, We define

$$\bar{a} = \bar{b} \text{ iff } a^- = b^-, a^+ = b^+;$$

$$\bar{a} \leq \bar{b} \text{ iff } a^- \leq b^-, a^+ \leq b^+;$$

$$\bar{a} < \bar{b} \text{ iff } \bar{a} \leq \bar{b}, \text{ and } \bar{a} \neq \bar{b};$$

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Clearly, $([I], \leq)$ constitutes a partial order set with a minimal element, $\bar{0} = [0, 0]$ and a maximal element $\bar{1} = [1, 1]$.

Definition 1.1 Let G be an ordinary group, mapping $\bar{A} : G \rightarrow [I]$ is called an interval valued fuzzy set on group G . Let $\bar{A}(x) = [A^-(x), A^+(x)]$, $A^-(x) \leq A^+(x)$, for any $x \in G$. Then the ordinary fuzzy set $A^- : G \rightarrow I$ and $A^+ : G \rightarrow I$ is called a lower fuzzy set and an upper fuzzy set of \bar{A} respectively.

Let $IF(G)$ denote the family of all of interval valued fuzzy sets on group G . For any $x \in G$, define $\Phi(x) = [0, 0]$, $G(x) = [1, 1]$, evidently, $\Phi, G \in IF(G)$.

In addition, for every $[\lambda_1, \lambda_2] \in [I]$, $\bar{A} \in IF(G)$. Let $\bar{A}_{[\lambda_1, \lambda_2]} = \{x \in G \mid A^-(x) \geq \lambda_1, A^+(x) \geq \lambda_2\}$. Then $\bar{A}_{[\lambda_1, \lambda_2]}$ is called a $[\lambda_1, \lambda_2]$ -level set of \bar{A} . Obviously, $\bar{A}_{[\lambda_1, \lambda_2]} = A_{\lambda_1}^- \cap A_{\lambda_2}^+$.

Definition 1.2 Let T be an ordinary T -norm, if for arbitrary $a \in I$, we have $T(a, a) = a$, then T is called an idempotent norm.

Definition 1.3 Let T be an idempotent norm, mapping $T_H : [I] \times [I] \rightarrow [I]$, for any $\bar{a}, \bar{b} \in [I]$, $T_H(\bar{a}, \bar{b}) \triangleq [T(a^-, b^-), T(a^+, b^+)]$. Then T_H is said to be an idempotent interval norm.

By the definition of the order in $[I]$, clearly, T_H satisfies every properties which an ordinary T -norm has, and for any $\bar{a} \in [I]$, $T_H(\bar{a}, \bar{a}) = \bar{a}$.

2 The homomorphism of the T_H -interval valued fuzzy subgroups.

Definition 2.1 Let G be a group, T_H be an idempotent interval norm, $\bar{A} \in IF(G)$, if for arbitrary $x, y \in G$, $\bar{A}(x \cdot y) \geq T_H(\bar{A}(x), \bar{A}(y))$. Then we call \bar{A} a T_H -interval valued fuzzy extensive group on G .

Definition 2.2 Let \bar{A} be a T_H -interval valued fuzzy extensive group on group G , If for any $x \in G$, $\bar{A}(x^{-1}) \geq \bar{A}(x)$, then we call \bar{A} a T_H -interval valued fuzzy subgroup on G .

Let $IF(G, T_H)$ denote the family of all of the T_H -interval valued fuzzy subgroups on G .

Definition 2.3 Let X and Y be two ordinary sets, mapping $f : X \rightarrow Y$ induces two mappings $F_f : IF(X) \rightarrow IF(Y)$ and $F_f^{-1} : IF(Y) \rightarrow IF(X)$, where $IF(Z) = \{\bar{A} \mid \bar{A} : Z \rightarrow [I]\}$, and $Z = X, Y$.

$$\text{We define } F_f(\bar{A})(y) = \begin{cases} \sup_{x \in f^{-1}(y)} \bar{A}(x) & f^{-1}(y) \neq \emptyset \text{ and } y \in Y \\ [0, 0] & \text{otherwise} \end{cases}$$

$$F_f^{-1}(\bar{B})(x) = \bar{B}(f(x)), x \in X.$$

Where $\bar{A} \in IF(X)$, $\bar{B} \in IF(Y)$, $f^{-1}(y) = \{x \in X \mid f(x) = y\}$.

Then mapping F_f and F_f^{-1} is called an interval valued fuzzy transformation and its inverse transformation induced by f respectively. This is a process which the mapping between interval valued fuzzy sets is transformed by an ordinary mapping f . We call the process the extension principle of interval valued fuzzy sets.

For arbitrary $\bar{A} \in IF(X)$, $\bar{B} \in IF(Y)$, clearly, we have

$$F_f(\bar{A})(y) = [\bigvee_{x \in f^{-1}(y)} A^-(x), \bigvee_{x \in f^{-1}(y)} A^+(x)] = [F_f(A^-)(y), F_f(A^+)(y)]$$

$$F_f^{-1}(\bar{B})(x) = [B^-(f(x)), B^+(f(x))] = [F_f^{-1}(B^-)(x), F_f^{-1}(B^+)(x)]$$

Where $F_f(A^-)$, $F_f(A^+)$, $F_f^{-1}(B^-)$ and $F_f^{-1}(B^+)$ obey the extension principle of ordinary fuzzy sets.

Theorem 2.1 Let G and \bar{G} be extensive groups, T_H be an idempotent interval norm, $f : G \rightarrow \bar{G}$ be a surjective homomorphism \bar{A} be a T_H - interval valued fuzzy extensive group on G . Then $F_f(\bar{A})$ is a T_H - interval valued fuzzy extensive group on \bar{G} .

Proof Otherwise, there exists $y_1, y_2 \in \bar{G}$ such that

$$F_f(\bar{A})(y_1 \cdot y_2) < T_H(F_f(\bar{A})(y_1), F_f(\bar{A})(y_2))$$

Since f is a surjective, and by the definition of the supremum of the elements in $[I]$, We can know that there exists corresponding $x_1, x_2 \in G$ such that $f(x_1) = y_1$, $f(x_2) = y_2$ and

$$F_f(\bar{A})(y_1 \cdot y_2) < T_H(\bar{A}(x_1), \bar{A}(x_2))$$

Furthermore, As f is an homomorphic mapping, $f^{-1}(y) \neq \emptyset$.

Then, $f(x_1 \cdot x_2) = f(x_1) \cdot f(x_2) = y_1 \cdot y_2$

Consequently, we have $F_f(\bar{A})(y_1 \cdot y_2) = \sup_{f(x)=y_1 \cdot y_2} \bar{A}(x) \geq \bar{A}(x_1 \cdot x_2)$

Therefore $\bar{A}(x_1 \cdot x_2) < T_H(\bar{A}(x_1), \bar{A}(x_2))$

This contradicts that \bar{A} is a T_H - interval valued fuzzy extensive group on G .

Theorem 2.2 Let G and \bar{G} be extensive groups, T_H be an idempotent interval norm, $f: G \rightarrow \bar{G}$ be a surjective homomorphism, \bar{B} be a T_H - interval valued fuzzy extensive group on \bar{G} . Then $F_f^{-1}(\bar{B})$ is a T_H - interval valued fuzzy extensive group on G , too.

Proof For each $x_1, x_2 \in G$, We obtain easily

$$\begin{aligned} F_f^{-1}(\bar{B})(x_1 \cdot x_2) &= \bar{B}(f(x_1 \cdot x_2)) = \bar{B}(f(x_1) \cdot f(x_2)) \\ &\geq T_H(\bar{B}(f(x_1)), \bar{B}(f(x_2))) \\ &= T_H(F_f^{-1}(\bar{B})(x_1), F_f^{-1}(\bar{B})(x_2)) \end{aligned}$$

Thus, the theorem is proved.

As to the homomorphism of groups, we have the following theorem.

Theorem 2.3 Let G and \bar{G} be groups, $f: G \rightarrow \bar{G}$ is a surjective homomorphism. If $\bar{A} \in IF(\bar{G}^S, T_H)$. Then $F_f(\bar{A}) \in IF(G^S, T_H)$.

Proof From theorem 2.1, for any $y \in \bar{G}$, we need only prove that

$$F_f(\bar{A})(y^{-1}) \geq F_f(\bar{A})(y).$$

$$\begin{aligned} \text{Actually, } F_f(\bar{A})(y^{-1}) &= \sup_{x \in f^{-1}(y^{-1})} \bar{A}(x) = \sup_{f(x)=y^{-1}} \bar{A}((x^{-1})^{-1}) \\ &\geq \sup_{f(x^{-1})=y} \bar{A}(x^{-1}) = \sup_{f(z)=y} \bar{A}(z) \\ &= f(\bar{A})(y) \end{aligned}$$

Hence, $F_f(\bar{A}) \in IF(G^S, T_H)$.

Theorem 2.4 Let G and \bar{G} be ordinary groups, $f: G \rightarrow \bar{G}$ be a surjective homomorphism. If $\bar{B} \in IF(\bar{G}^S, T_H)$, then $F_f^{-1}(\bar{B}) \in IF(G^S, T_H)$.

Proof For arbitrary $x \in G$, We need only verify that

$$F_f^{-1}(\bar{B})(x^{-1}) \geq F_f^{-1}(\bar{B})(x)$$

$$\begin{aligned} \text{In fact } F_f^{-1}(\bar{B})(x^{-1}) &= \bar{B}((f(x^{-1}))) = \bar{B}((f(x))^{-1}) \\ &\geq \bar{B}(f(x)) = F_f^{-1}(\bar{B})(x) \end{aligned}$$

And so, $F_f^{-1}(\bar{B}) \in IF(G^S, T_H)$.

Note Let mapping $f: G \rightarrow \bar{G}$ be an homomorphism of groups, denote

$$\ker f = \{x \in G \mid f(x) = \bar{e}\}$$

Where \bar{e} is the identity element of \bar{G} . We call $\ker f$ the kernel of f .

Theorem 2.5 Let mapping $f: G \rightarrow \bar{G}$ be an homomorphism of groups, $\bar{A} \in IF(G^s, T_H)$, e be the identity element of group G , and $\bar{A}(e) = [1, 1]$. Then for any $y \in \ker f$, $\bar{A}(y) = \bar{A}(e)$ iff $\bar{A}(x \cdot y) = \bar{A}(x)$ for all $x \in G$, $y \in \ker f$.

Proof Necessity. For each $x \in G, y \in \ker f$, by the known conditions, we can derive from that

$$\begin{aligned}\bar{A}(x \cdot y) &\geq T_H(\bar{A}(x), \bar{A}(y)) = T_H(\bar{A}(x), \bar{A}(e)) \\ &= [T(A^-(x), 1), T(A^+(x), 1)] \\ &= [A^-(x), A^+(x)] \\ &= \bar{A}(x)\end{aligned}$$

On the other hand, since f is an homomorphism of groups. Then we get

$$f(y^{-1}) \cdot f(y) = f(y^{-1} \cdot y) = f(e) = \bar{e}$$

Where \bar{e} is the identity element of \bar{G} .

Therefore, $f(y^{-1}) = (f(y))^{-1} = \bar{e}^{-1} = e$. i.e., $y^{-1} \in \ker f$

Consequently, $\bar{A}(y^{-1}) = \bar{A}(e)$

Thus, we can infer that

$$\begin{aligned}\bar{A}(x) &= \bar{A}(x \cdot y \cdot y^{-1}) \geq T_H(\bar{A}(x \cdot y), \bar{A}(y^{-1})) \\ &= T_H(\bar{A}(x \cdot y), \bar{A}(e)) \\ &= [T(A^-(x \cdot y), 1), T(A^+(x \cdot y), 1)] \\ &= [A^-(x \cdot y), A^+(x \cdot y)] \\ &= \bar{A}(x \cdot y)\end{aligned}$$

Hence, $\bar{A}(x \cdot y) = \bar{A}(x)$

Sufficiency. Take $x = e \in G$, for arbitrary $y \in \ker f$.

Evidently, we have

$$\bar{A}(y) = \bar{A}(e \cdot y) = \bar{A}(e)$$

Corollary Let $f: G \rightarrow \bar{G}$ be an isomorphism of groups. $\bar{A} \in IF(G^s, T_H)$. Then $F_f^{-1}(F_f(\bar{A})) = \bar{A}$.

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