# **CONTRIBUTION OF SLOVAKIA IN THE FUZZY SETS THEORY**

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#### 1. Introduction

In the more then 30 years old history of the theory of fuzzy sets, thousands of papers in this domain have been published all over the world. In the last ten years, also Slovakia has added several valuable contributions. The aim of this paper is to give a (surely not exhaustive) overview of Slovak "fuzzy" results.

The beginning of Slovak research on fuzzy sets theory goes back to the first European fuzzy workshop in Warszawa in 1986, where the concept of fuzzy quantum spaces was introduced by B. Riečan and A. Dvurečenskij [103], and the fuzzy connectives in the framework of expert systems were discussed by R. Mesiar [93]. Later, J. Bán [8,9] started investigating the fuzzy valued mappings. The real starting point for a wide group of Slovak fuzzy researchers was the first big international conference (organized in the former Czechoslovakia) held in Bechyně ,1990, which was coorganized by Riečan and Mesiar. Since 1992, the regular biannual international conferences are organized in Liptovský Ján by Military Academy in Liptovský Mikuláš, under the scientific direction of E. P. Klement from Austria. B. Riečan and R. Mesiar. Actually, there are several topics investigated by Slovak fuzzy researchers. Well known is the Slovak school on fuzzy quantum structures (Dvurečenskij, Riečan, Chovanec, Kôpka, etc.) with several remarkable results in the MV algebras domain. Interesting results were obtained in the area of fuzzy connectives, especially in the triangular norms theory (Mesiar, Šabo, Viceník, Smutná). Another remarkable area includes the calculus of fuzzy numbers (Mesiar, Kolesárová, Marková-Stupňanová, Harman) and the area of fuzzy linear algebra (Cechlárová, Gavalec). Most recently, the fuzzy analysis is investigated (Kalina, Janiš). Note that several recent results in the fuzzy domain were presented by Slovak researchers at the 7th world congress IFSA97 in Prague which was coorganized by them jointly with the Czech colleaguer. In the next sections, we briefly recall some results from the mentioned areas.

## 2. Fuzzy quantum structures

The domain of fuzzy quantum structures has been intensively investigated by Riečan, Dvurečenskij, Chovanec, Kôpka, Jurečková, Rybáriková, Markechová, Tirpáková, Mesiar, Kolesárova, etc. Several analogons of classical results were obtained (e.g. limit theorems). The structure of fuzzy quantum spaces was characterized by Mesiar in [66] and by Dvurečenskij in [19] and thus making previously obtained results more transparent or easier provable. Similarly, the characterization of fuzzy quantum logics [72, 77] by Mesiar gave a complete picture of the domain.

An important concept of fuzzy difference poset was introduced by Kôpka in 1992 (at the first famous FSTA conference) and later developed and generalized (under the name difference poset) by Kôpka and Chovanec and by many other domestic and foreign scientists from the area of quantum structures. An exhaustive overview of the mentioned results can be found in the monograph of Riečan and Neubrunn [106]. Recently, the efforts of the mentioned group in the MV algebras investigation gave rise to several important results on the measure theory on MV algebras, see e.g. Riečan [103].

#### 3. Fuzzy connectives

The starting point of the fuzzy connectives research in Slovakia came from the investigation of the propagation of uncertainty in the expert systems by Mesiar. An efficient help of E. P. Klement for several Slovak scientists (literature, lectures in Slovakia, stages of Slovak researchers in Linz) enabled to obtain several deep results on triangular norms (Mesiar, Viceník, Smutná) and related operators (Šabo, Komorníková, Kolesárová, Mesiar).

Recall, e.g., the characterization of the fuzzy sigma-algebras based on fundamental triangular norms and of the corresponding measures by Mesiar [67, 70] and Mesiar with Navara [89]. Among others, note the construction methods for t-norms by means of pseudo-inverses of monotone functions, where Viceník [116] showed that pseudo-inverse need not be a quasi-inverse (which was believed to be true till Viceník's paper appeared). Smutná [108] showed that the strictness of a t-norm need not imply the Archimedean property and the Archimedean property need not result to the relative cancellativity (T(x,y) = T(x,z)) iff T(x,y)(y-z) = 0, see [109]. As a continuation of the work of Klement, Mesiar and Pap, Viceník [115] showed that an additive generator may generate a t-norm possessing even a non-trivial idempotent element. New class of aggregation operators based on triangular norms was proposed by Kolesárová and Komorníková [41]. Several interesting results on implicators may be found in papers of Šabo [111] and Mesiar (also with DeBaets) [3,74]. The limit and approximation properties of t-norms important for practical applications were studied by Mesiar in [86,87]. Note that 7 papers written by Slovak researchers (jointly with foreign coworkers) will appear in a special issue of Fuzzy Sets and Systems devoted to triangular norms which is just in press (e.g., 41, 90, 116).

## 4. Fuzzy numbers

Triangular norm-based calculus of fuzzy numbers seems to be one of the favorite Slovak topics. The first pioneer in the domain was Harman [26]. Among the most important results recall the transformation principle by Mesiar [80, 82] which allowed to improve several previous results (e.g. of Fuller, Hwong, etc.), but also to solve some philosophical problems (e.g., what type of fuzzy numbers shape is preferable). The fuzzy arithmetics was intensively studied in the works of Mesiar, Marková-Stupňanová and Kolesárová. A series of improvements and generalizations of results on analytical expressions of t-norm based additions of fuzzy numbers was closed in [6, 58, 83]. Idempotents for fuzzy addition based on a t-norm T [59, 60] are important also in the pseudo-analysis of E. Pap (as idempotents of pseudo-convolutions). The preserving of shapes [38, 39, 40, 82] allows to reduce the computation with fuzzy quantities to the computation with parameters, similarly as in some special cases in the probability theory. Most recently, new types of fuzzy quantities (generated fuzzy quantities) were proposed and studied by Mareš, Mesiar (and DeBaets), see [5, 48].

#### 5. Conclusions

We have briefly mentioned some of interesting and valuable results from the fuzzy set theory area obtained by Slovak researchers. These and several other results are published in the papers given in the next list of references. It was not the aim of this paper to describe all Slovak results in the domain (what is, in fact, impossible) but to give an impression of the work in the fuzzy area in Slovakia. All these results form the theoretical background of the fuzzy set theory. However, though there are also several applications of the fuzzy set theory, see e.g. [108], real industrial applications are still missing. Nevertheless, we hope that this lack will be any soon not true, as far as the first contacts with the industry have been already opened.

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