LINEARITY OF FUZZY FUNCTIONS DERIVATIVES

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ABSTRACT. A graph of a real fuzzy function can be considered as a collection of nonintersecting graphs of real functions (level functions), connecting values with the same membership degree. It is possible to define a derivative of a fuzzy function in terms of derivatives of its level functions. This derivative at a given point is a fuzzy real number. We show that if we use the triangular norm T for addition of fuzzy functions and their derivatives, then we obtain the inequality $(f +_T g)' \leq f' +_T g'$. An example shows that in general the equality does not hold here.

We will deal with a real fuzzy function of a real (crisp) variable, i.e. with a function that assigns a fuzzy number to a real number. First let us consider a fuzzy number as an LR-fuzzy number (for more details see [4]) with both shape functions strictly monotone. The addition of LR-fuzzy numbers based on Zadeh's extension principle is studied in [2] and [3]. We can define *level functions* for a fuzzy function in the following way:

If $\alpha \in (0;1)$, then the level function f_α of a fuzzy function f is a real function for which $f_\alpha(x) = y$ if and only if $f(x)(y) = \alpha$ and y belongs to the decreasing part of $f(x)$ (i.e. it is greater than the peak of $f(x)$). If $\alpha \in (-1,0)$, then $f_\alpha(x) = y$ if and only if $f(x)(y) = -\alpha$ and y belongs to the increasing part of $f(x)$ (i.e. it is less than the peak of $f(x)$). Finally, by f_1 we denote a real function assigning the peak of $f(x)$ to x.

In [1] a derivative of a fuzzy function is defined, using its level functions. We will briefly recall the definition of this derivative:

Suppose the fuzzy function f is defined at a point a and suppose each its level function is continuously differentiable at a. Then its derivative $f'(a)$ is the fuzzy number with the following property: if $\alpha \in (0;1)$, then the α-cut of $f'(a)$ is the interval $(I;S)$, where

$$I = \inf\{f'_\beta(a); |\beta| \geq \alpha\}$$

and

$$S = \sup\{f'_\beta(a); |\beta| \geq \alpha\}.$$

It can happen that I or S or both have the infinite value, hence the interval $(I;S)$ may be unbounded.

The interval $(I;S)$ is in fact not an α-cut, but a strict α-cut, but as throughout the whole paper we use only this type of cuts, we will omit the word "strict" and

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use just the term "cut". The α-cut of a fuzzy number A will be denoted by the symbol $[A]_\alpha$.

Given two fuzzy functions f, g, both differentiable at a a question of the linearity arises. In other words we ask whether it holds $(f +_T g)'(a) = f'(a) +_T g'(a)$ for a given t-norm, if both sides exist. The following proposition claims that one inequality always holds and an example will show that the opposite one in general does not hold.

Proposition 1. *If the fuzzy functions f and g have fuzzy derivatives f' and g' at the point a, their sum $f +_T g$ has the fuzzy derivative $(f +_T g)'$ at a, where T is a t-norm, then $(f +_T g)'(a) \leq f'(a) +_T g'(a)$.*

Proof. Suppose $0 < \alpha < 1$. We will show that the α-cut of the fuzzy number $(f +_T g)'(a)$ is a subset of the α-cut of the fuzzy number $f'(a) +_T g'(a)$.

Let $x \in [(f +_T g)']_\alpha$. Then there exist $\beta, \gamma, |\beta| \geq \alpha, |\gamma| \geq \alpha$ such that

$$h_\beta(a) < x < h_\gamma(a),$$

where h_β and h_γ are level functions of the fuzzy function $(f +_T g)'$. Due to the assumption of continuous differentiability of the level functions there are level functions H_β, H_γ of the fuzzy function $f +_T g$ for which

$$(H_\beta)'(a) = h_\beta(a), \quad (H_\gamma)'(a) = h_\gamma(a).$$

From the extension principle we obtain the following: If A, B are fuzzy numbers and $\delta \in (0, 1]$, then

$$[A +_T B]_\delta = \cup \{[A]_\rho + [B]_\sigma; T(\rho, \sigma) \geq \delta\}.$$

This enables us to claim that there are level functions $f_{\gamma_1}, f_{\beta_1}$ of f and $g_{\gamma_2}, g_{\beta_2}$ of g such that

$$H_\beta(a) = f_{\beta_1}(a) + g_{\beta_2}(a), \quad T(\beta_1, \beta_2) \geq \alpha,$$

and

$$H_\gamma(a) = f_{\gamma_1}(a) + g_{\gamma_2}(a), \quad T(\gamma_1, \gamma_2) \geq \alpha.$$

As the chosen element x fulfills the inequalities

$$f'_{\gamma_1}(a) + g'_{\gamma_2}(a) \leq x \leq f'_{\beta_1}(a) + g'_{\beta_2}(a)$$

using again the above mentioned equality we see that x belongs to the α-cut of the sum $f'(a) +_T g'(a)$ what completes the proof. $\square$

The opposite inequality $f'(x) + g'(x) \leq (f(x) + g(x))'$ in general does not hold. In the following example we use the minimum t-norm for the addition (i.e. $T_{min}(x, y) = \min\{x, y\}$) and show that the inequality from Proposition 1 turns to be sharp.

Example 1. Let the fuzzy functions $f, g : [0;1] \to R$ be given by the following formulas: For $x \in [0;1]$ put

$$f(x)(t) = \max\left\{0; 1 - \frac{|t|}{x+1}\right\}, \quad t \in R,$$

$$g(x)(t) = \max\left\{0; 1 - \frac{|t|}{2-x}\right\}, \quad t \in R.$$

All the level functions of f and g are linear and hence differentiable on the interval $[0;1]$. For an arbitrary $x \in [0;1]$ the derivatives of f and g are equal (we take the one-side derivatives at the endpoints of the interval) and

$$f'(x)(t) = g'(x)(t) = \max\{0; 1 - |t|\}, \quad t \in R.$$

We see that both f' and g' are constant fuzzy functions on $[0;1]$. Note that their sum with respect to the norm T_{min} is again a constant fuzzy function and its common value is a fuzzy number $\max\left\{0; 1 - \frac{|t|}{2}\right\}$, $t \in R$.

On the other hand the sum $f +_{T_{min}} g$ is a constant fuzzy function with the common value

$$(f +_{T_{min}} g)(x)(t) = \max\left\{0; 1 - \frac{|t|}{3}\right\}, \quad t \in R,$$

for each $x \in [0,1]$. Hence all the level functions are constant and therefore for all $x \in [0,1]$ the value $(f +_{T_{min}} g)'(x)$ is the crisp number zero. Thus we obtain

$$(f +_{T_{min}} g)'(x) < f'(x) +_{T_{min}} g'(x),$$

which shows that the derivative defined in [1] is not additive.

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