ON A TYPE OF FUZZY CONTINUITY

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ABSTRACT. This paper is a continuation of [1]. It presents one possible approach to fuzzy continuity of real functions, utilizing the concept of a nearness. The connection between standard continuity and N-fuzzy continuity is investigated.

1. INTRODUCTION

In many publications in fuzzy analysis there are presented various approaches to the fuzzification of concepts like distance, metric and topology.

These consequently enable to fuzzify convergence, continuity, differentiability and other related notions (see, for example, [2],[3],[4],[5]).

The aim of this paper is to introduce one possible and quite natural fuzzification of continuous functions based on a "nearness" of real numbers.

In general the nearness of elements of any set X is considered to be a fuzzy relation N on X with some "appropriate" properties. These properties are not stable, but depend on the concrete problem and, of course, on the structure of X. In the majority it is accepted that $N(x,x) = 1$ for each $x \in X$ and $N(x,y) = N(y,x)$ for each $x, y \in X$. Further properties are usually connected with the structure of X.

If X is a set without any structure, then it is natural to require moreover a property substituting in some sence triangular inequality. (Questions relevant to this problem will be treated in this article.)

On the other hand, the presence of algebraical, topological or lattice structure makes it possible to require further reasonable and natural properties.

We will restrict ourselves to the real case, i.e. $X = \mathbb{R}$ - the set of all real numbers.

With respect to the algebraical operations, topology and linear ordering of real numbers, there are several possible ways, how to define a nearness on $\mathbb{R}$.

In [1] there is the following definition:

A function $N : \mathbb{R} \times \mathbb{R} \to [0,1]$ is called a shift-invariant nearness, if

(1) $N(x,x) = 1$ for each $x \in \mathbb{R}$
(2) $N(x,y) = N(y,x)$ for each $x, y \in \mathbb{R}$

Key words and phrases. Nearness, nearness-generating function, N-fuzzy continuity.
This research has been supported by grant VEGA 95/5305/471.

(3) $N(x,y) \geq N(x,z)$ for each $x,y,z \in \mathbb{R}$, such that $x \geq y \geq z$ or $x \leq y \leq z$
(4) $\lim_{n\to\infty} x_n = \infty \Longrightarrow \lim_{n\to\infty} N(x_n, x_0) = 0$ for each $x_0 \in \mathbb{R}$
(5) $N(x,y) = N(x+z, y+z)$ for each $x,y,z \in \mathbb{R}$

Another definition of a nearness on $\mathbb{R}$ is in [5]:

A function $N : \mathbb{R} \times \mathbb{R} \to [0,1]$ is said to be a uniform nearness, if there exists a non-increasing function $b : [0,\infty] \to [0,1]$ such that $b(0) = 1$, $\lim_{x\to\infty} b(x) = 0$ and $N(x,y) = b(|x-y|)$ for each $x,y \in \mathbb{R}$.

The main result of [1] is the assertion:

A nearness is shift invariant if and only if it is uniform.

2. Non-uniform nearnesses

The requirement of uniformity for a nearness of real numbers, expressed, in fact, by the property (5) is quite natural, but not always necessary.

The geometrical meaning of (5) is the following:

$N(x,y)$, as a function of two variables, is constant on lines parallel to the line $x = y$.

Consider now a nearness defined as follows: $N(x,y) = 1$, if $x \geq 0, y \geq 0$ and $y \in [\frac{x}{2}, 2x]$, or if $x \leq 0, y \leq 0$ and $y \in [2x, \frac{x}{2}]$ and $N(x,y) = 0$ else.

It can be shown, that N satisfies the properties (1)-(4), but not the property (5). For example:

$N(1, \frac{1}{4}) \neq N(1+1, \frac{1}{4}+1)$, therefore N is not shift-invariant and thus neither uniform.

One of the consequences of this fact is that a fuzzy local nearness $N_{x_0}(x)$, which can be defined for any nearness and any point $x_0 \in \mathbb{R}$ by the formula

$$N_{x_0}(x) = N(x_0, x) \text{ for each } x \in \mathbb{R}$$

needn't be always a fuzzy interval, symmetrical to the line $x = x_o$ (see [1]) and if x_1, x_2 are two different real numbers, than the graph of $N_{x_2}(x)$ one cann't obtain by shifting the graph of $N_{x_1}(x)$:

Example. Let $N(x,y)$ be defined as above. Let $x_1 = 1$, $x_2 = 2$. Then

$N_1(x) = 1$, *for* $x \in [\frac{1}{2}, 2]$ *and* $N_1(x) = 0$ *else.*

$N_2(x) = 1$, *for* $x \in [1,4]$ *and* $N_2(x) = 0$ *else.*

Obviously, none of their graphs is symmetrical with the centre at $x = 1$ or $x = 2$, resp.

3. N-fuzzy continuity

The concept of a nearness N in $\mathbb{R}$ enables to introduce a type of convergence with respect to N (see Definition 2 in [1]) and consequently a type of continuity for real functions of a real variable.

From now on we tacitly assume that N is always a uniform nearness on $\mathbb{R}$.(Despite the following definitions and notions make sence also in a more general case.)

That means: $N(x,y) = b(|x-y|)$, for a function b mentioned above. The function b is called a nearness-generating function and is determined uniquely (see [1]).

Definition 1. Let N be a nearness , f be a real function of a real variable and $x_0 \in \mathbb{R}$. The function f is said to be N-fuzzy continuous (briefly N-continuous) at x_0 if for any number $\epsilon < 1$ there exists a number $\delta < 1$ such that for each $x \in \mathbb{R}$: $N(x_o, x) > \delta$ implies $N(f(x_0), f(x)) > \epsilon$.

Theorem 1. *Let f be a real function of a real variable. The function f is N-continuous at a point $x_0 \in \mathbb{R}$ if and only if for any sequence $\{x_n\}$ of real numbers $\lim_{n\to\infty} N(x_n, x_0) = 1$ implies $\lim_{n\to\infty} N(f(x_n), f(x_0)) = 1$.*

Proof. We begin by proving the sufficient condition:

Let us suppose that there exists $\epsilon_0 < 1$ such that for each natural number n there is $x_n \in \mathbb{R}$ such that $N(x_0, x_n) > 1 - \frac{1}{n}$ and simultaneously $N(f(x_0), f(x_n)) \leq \epsilon_0$. Then $\lim_{n\to\infty} N(x_0, x_n) = 1$, but $\lim_{n\to\infty} N(f(x_0), f(x_n)) \neq 1$.

The necessary condition:

Let for each $\epsilon < 1$ there exists $\delta(\epsilon) < 1$ such that

$$N(x_0, x) > \delta \Longrightarrow N(f(x_0), f(x)) > \epsilon$$

for any real x.

Suppose $\lim_{n\to\infty} N(x_0, x_n) = 1$ and $\epsilon_0 < 1$ is arbitrary, but fixed. Put $\delta_0 = \delta(\epsilon_0)$, hence there exists a number n_0 such that for any natural n, $n > n_0 : N(x_0, x_n) > \delta_0$ and therefore $N(f(x_0), f(x_n)) > \epsilon_0$.

It is evident that Theorem 1 is not affected if we delete uniformity of the nearness.

If X is a set of real numbers and f is a real function defined on X then, as usual, f is called N-continuous on X, if it is N-continuous at each $x \in X$.

Definition 2. Let N be a nearness and X be a set of real numbers. A real valued function f defined on X is called uniformly N-continuous on X, if for any $\epsilon < 1$ there exists $\delta < 1$ such that for all $x, y \in X$, $N(x,y) > \delta$ implies $N(f(x), f(y)) > \epsilon$.

It is evident, that a family of N-continuous (at a point, or on a set) functions depends essentially on N and its properties.

Theorem 2. *Let N be a nearness with a nearness-generating function b satisfying the following condition*

(*) $\quad x_n \to 0 \Longleftrightarrow b(x_n) \to 1.$

Then a real function of a real variable f is continuous at a point $x_0 \in \mathbb{R}$ if and only if it is N-continuous at x_0.

Proof. The assertion follows immediately from (*) and the fact that $N(x,y) = b(|x-y|)$. We outline the proof only for the "if" part; the other part is left to the reader:

$|x_n - x_0| \to 0 \Longleftrightarrow b(|x_n - x_0|) = N(x_n, x_0) \to 1 \quad \Longrightarrow N(f(x_n), f(x_0)) = b(|f(x_n) - f(x_0)|) \to 1 \Longleftrightarrow |f(x_n) - f(x_0)| \to 0$

(*) is fulfilled evidently if b is moreover decreasing and continuous, for example if

$$b(x) = \frac{1}{1+x}$$

but also if

$b(x) = 1 - x$ for $x \in [0, \frac{1}{2}]$ and $b(x) = 0$ for $x > \frac{1}{2}$, which is neither continuous nor decreasing. This example indicates that N, and therefore also N-continuity, depends mostly on the behaviour of b in a right neighbourhood of zero.

Corollary. *Let N be a nearness as in Theorem 1, let X be a set of real numbers.*

Then a real function defined on X is (uniformly) continuous on X if and only if it is (uniformly) N-continuous on X.

Theorem 3. *Let N be a nearness with such a nearness-generating function b that there exists a number $K \in (0,1)$ such that $b(x) < K$ for all $x > 0$.*

Then any real function of a real variable is uniformly N-continuous on any set of real numbers.

Proof. Let $X \subset \mathbb{R}$ and $\epsilon < 1$. Then if $\delta \in (K, 1)$ is arbitrary and $x, y \in X$ are such that $N(x,y) > \delta$, it follows that $b(|x-y|) > \delta > K$ and therefore $|x - y| = 0$, hence $x = y$ and thus $N(f(x), f(y)) = 1 > \epsilon$

So this kind of nearness-generating functions gives trivial N-continuous functions. It follows from the fact, that in this case N-convergent are only sequences of real numbers which are constant from some term (see [1]).

Typical relevant nearness-generating functions are:

$b(x) = 0$ if $x > 0$ and $b(0) = 1$ or

$b(x) = \frac{1}{2+x}$ if $x > 0$ and $b(0) = 1$.

In both discussed cases $b(x) = 1$ implies $x = 0$.

Now let's consider an opposite situation. Suppose that there exists a number $a > 0$ such that $b(x) = 1$ for $x \in [0, a]$.

In fact it means, that the corresponding nearness distinguishes only points having their distance greater than a. N-continuity, obtained in this case, is a little unusual, but in some sence more expressing an intuitive comprehension of continuity (not requiring continuous graph):

If two arguments of a function are "near" to each other, then values of the function at these arguments are "near" to each other, as well.

Example. Let $0 < a < c, a, c \in \mathbb{R}$. Define a nearness-generating function b as follows:

$b(x) = 1$ if $x \in [0, a)$, $b(x) = \frac{c-x}{c-a}$ if $x \in [a, c]$, $b(x) = 0$ if $x \in (c, \infty)$.

It means that the corresponding nearness N is continuous, $N(x,y) = 1$ if $|x-y| < a$, $N(x,y) = 0$ if $|x - y| > c$ and on the interval $[a, c]$ N is linearly decreasing.

Now, let a_1, c_1 be such real numbers that $0 < a_1 < a$ and $c_1 > \frac{c}{2}$. Consider now a function $f : \mathbb{R} \to \mathbb{R}$:

$f(x) = na_1$ if $x \in [nc_1, (n+1)c_1)$ for each $n \in Z$.

It can be easily proved that f is uniformly N-continuous on $\mathbb{R}$, despite it has infinitely many points of discontinuity.

To see this, let $\epsilon < 1$; we can take an arbitrary $\delta \in (0,1)$, because $N(x,y) > \delta > 0$ implies $|x - y| < c < 2c_1$, what implies

$|f(x) - f(y)| \leq a_1 < a$, and therefore $N(f(x), f(y)) = 1 > \epsilon$.

On the other hand, with respect to this nearness, even continuous functions needn't be N-continuous.

For example the function $f(x) = x^2$ is not N- continuous at any point x_0 such that $|x_0 + \frac{a}{2}| > \frac{1}{2}$.

Let's sketch a trivial verification :

$N(x_0, x_0 + a + \frac{1}{n}) = b(|x_0 - x_0 - a - \frac{1}{n}|) = b(|a + \frac{1}{n}|) \to b(a) = 1$, but

$N(x_0^2, (x_0 + a + \frac{1}{n})^2) = b(|x_0^2 - (x_0 + a + \frac{1}{n})^2|) \to b(|2x_0 a + a^2|) \neq 1$.

From the previous example it follows immediately that if a nearness N doesn't distinguish some points with positive distance, then there always exist continuous functions, which are not N-continuous.

References

[1] Dobrakovová J., *On a fuzzy nearness*, Proceedings : Strojné inžinierstvo'98, vol. 1, pp. 33-37.

[2] Janiš V., *Fixed points of fuzzy functions*, Tatra Mountains Math. Publ. **12** (1997), 13-20.

[3] Janiš V., *Fuzzy uniformly continuous functions*, Tatra Mountains Math. Publ. **14** (1998), 177 - 180.

[4] Kalina M., *Derivatives of fuzzy functions and fuzzy derivatives*, Tatra Mountains Math. Publ. **12** (1997), 27-34.

[5] Kalina M., *On fuzzy smooth functions*, Tatra Mountains Math. Publ. **14** (1998), 153 - 159.