CLASS-THEORETICAL OPERATIONS OVER NONSTANDARD FUZZY UNIVERSE

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ABSTRACT. In this paper we give some basic principles of the syntax of nonstandard fuzzy sets. We use ultraproducts to construct our universe. Also the nonstandard fuzzy numbers and their arithmetic are introduced.

In sixties two new mathematical theories started developing – nonstandard analysis and fuzzy set theory. The nonstandard analysis is, in some sense, a comeback of the original Newton's infinitesimal calculus, i.e. (at the first glance) 'pure mathematics' – something being far away from the 'fuzzy world'. But this is just a matter of a propper interpretation. Namely, if you have a very long road, you enumerate the milestones and you put an observer there, he will have a horizon of his view beyond which he will not see anything. The numbers, corresponding to milestones not seen by our observer, you can consider to be infinite. In this connection the border between finite and infinite is of course vague, or – if you whish – fuzzy.

Similar fuzziness we can find also in the 'microworld'. Consider two different points, moving to each other. At some instant you will not be able to distinguish them. This distance you can consider to be infinitesimal. And the border between infinitesimal and noninfinitesimal is again vague – fuzzy.

So there is a natural connection between these two theories. This is an attempt to put them together and to show the possibility of modeling some kinds of fuzzy numbers in the nonstandard world. The author is aware of the fact that this is just the first attempt, and therefore shurely not the best one, anyhow, this topic is worth starting discussion on.

We will consider the ultraproduct construction to creating our universe.

We will start from the set $\mathbb{Q}$ of all rationals, over which we create our ultraproduct universe. For more details about the construction see, e.g., [1].

We get the system of all sequences of rational numbers. We say that the sequences $\{x_i\}_{i=1}^{\infty}$ and $\{y_i\}_{i=1}^{\infty}$ equal to each other at the level $[\alpha;\beta]$, or more precisely, the truth value of their equality is the interval $[\alpha;\beta]$, iff the following formula holds

$$\{x_i\}_{i=1}^{\infty} =_{[\alpha;\beta]} \{y_i\}_{i=1}^{\infty} \Leftrightarrow \left\{ \begin{array}{lcll} \alpha & = & \liminf\limits_{i\to\infty} & \dfrac{\operatorname{card}\{j \le i; x_j = y_j\}}{i}, \\ \beta & = & \limsup\limits_{i\to\infty} & \dfrac{\operatorname{card}\{j \le i; x_j = y_j\}}{i}. \end{array} \right.$$

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More generally, if we have a sequence of formulas with n free variables $\{\phi_i\}_{i=1}^{\infty}$, we say that the truth value of the formula $\{\phi_i\}_{i=1}^{\infty}$ is the interval $[\alpha;\beta]$ iff

$$\begin{aligned} \alpha &= \liminf_{i\to\infty} \frac{\mathrm{card}\{j \le i; \phi_j(x_j^1,\ldots,x_j^n)\}}{i}, \\ \beta &= \limsup_{i\to\infty} \frac{\mathrm{card}\{j \le i; \phi_j(x_j^1,\ldots,x_j^n)\}}{i}. \end{aligned} \tag{1}$$

In such a way we have constructed

(1) **standard** rational numbers, represented by constant sequences;
(2) **infinite** numbers – e.g., $\{n\}_{n=1}^{\infty}$ is greater than all standard rational numbers (represented by constant sequences), i.e. it is in fact infinite;
(3) **infinitesimal** numbers – e.g. $\{\frac{1}{n}\}_{n=1}^{\infty}$ is positive, but less than all positive standard rational numbers (let us note that 0 is also considered to be infinitesimal – the only standard infinitesimal number);
(4) last, but not least – we have constructed also **fuzzy** numbers – e.g., if

$$a_i = \begin{cases} 1 & \text{for even } i, \\ 2 & \text{for odd } i, \end{cases}$$

then $\{a_i\}_{i=1}^{\infty}$ is equal to 1 and at the same time to 2, both with the truth value one half.

Concerning the arithmetical operations, they are defined by Formula (1). Namely, if we have an operation $*$ given on the rational numbers, then this can be extended also to our 'ultraproduct numbers' in the following way

$$\{x_i\}_{i=1}^{\infty} * \{y_i\}_{i=1}^{\infty} \quad = \quad \{x_i * y_i\}_{i=1}^{\infty}.$$

The main goal of this paper is to present the class-theoretical operations over our universe.

We will distinguish objects (classes) of two categories – internal and external. **Internal** objects (also called sets) will be those definable by a sequence of formulas and **external** (also called proper classes) will be all other ones. E.g., all types of numbers, presented above, are internal objects.

Similarly to distinguishing of classes we will distinguish also class-theoretical operations (*i.e union, intersection and complementation*). We will have internal and external operations. The definition of the internal union, intersection and difference of two sets is straightforward. Let us have sets A, B defined by the sequences of formulas $\{\varphi_i\}_{i=1}^{\infty}$ and $\{\psi\}_{i=1}^{\infty}$, respectively. Let us use the notation $A = \{\varphi_i\}_{i=1}^{\infty}$ and $B = \{\psi_i\}_{i=1}^{\infty}$. Then their **internal union**, **internal intersection** and **internal difference** is defined by

$$A \cup B = \{\varphi_i \vee \psi_i\}_{i=1}^{\infty}; \quad A \cap B = \{\varphi_i \wedge \psi_i\}_{i=1}^{\infty}; \quad A \setminus B = \{\varphi_i \wedge \neg\psi_i\}_{i=1}^{\infty}.$$

But there is also another possibility for the definition of these operations. First, as a motivation, let us consider the following example

Example. Let us have sets A and B defined by the sequences of formulas $\{x_i = 1\}_{i=1}^{\infty}$ and $\{x_i = 2\}_{i=1}^{\infty}$, respectively.

Remind that, e.g., the sequence

$$a_i = \begin{cases} 1 & \text{for even } i, \\ 2 & \text{for odd } i, \end{cases} \tag{2}$$

is an element of both, A and B, at the level (with the truth value) $\frac{1}{2}$. Then the internal intersection $A \cap B$ is defined by the sequence of formulas

$$\{x_i = 1 \ \wedge \ x_i = 2\}_{i=1}^{\infty}$$

and hence $A \cap B = \emptyset$.

But as we can see in (2), there are elments, which belong both to A and B. And it can be interesting to know all such elements. So there is the need to have also 'intersections' (and, of course, all other class-theoretical operations) of some other type.

The most natural way to introducing the external versions of class-theoretical operations seems to be copying the 'usual' fuzzy sets (see, e.g. [2]). Namely to put

$$\begin{aligned} X \in_{[\alpha;\beta]} A \cap_T B &\Leftrightarrow \alpha = T(\alpha_1, \alpha_2) \ \& \ \beta = T(\beta_1, \beta_2), \\ X \in_{[\alpha;\beta]} A \cup_S B &\Leftrightarrow \alpha = S(\alpha_1, \alpha_2) \ \& \ \beta = S(\beta_1, \beta_2), \\ X \in_{[\alpha;\beta]} A \setminus_T B &\Leftrightarrow \alpha = T(\alpha_1, 1 - \beta_2) \ \& \ \beta = T(\beta_1, 1 - \alpha_2), \end{aligned}$$

where T is a triangular norm, S is the corresponding conorm and

$$X \in_{[\alpha_1;\beta_1]} A, \quad X \in_{[\alpha_2;\beta_2]} B.$$

These external class-theoretical operations can be used also for classes – external objects. Lastly, as the following example shows, the internal class-theoretical operations are not special cases of the external ones. Nevertheless, they are also important.

Example. Let A and B be sets defined by sequences of formulas

$$\{x_i \text{ is divisible by } 2\}_{i=1}^{\infty}, \quad \{x_i \text{ is divisible by } 3\}_{i=1}^{\infty},$$

respectively. Then $A \cap B$ is defined by

$$\{x_i \text{ is divisible by } 6\}_{i=1}^{\infty}.$$

Hence, the sequence

$$a_i = \begin{cases} 2 & \text{for even } i, \\ 3 & \text{for odd } i, \end{cases}$$

is an element both, of A and B, with the truth value $\frac{1}{2}$, but not of $A \cap B$. On the other hand, the sequence

$$a_i = \begin{cases} 1 & \text{for even } i, \\ 6 & \text{for odd } i, \end{cases}$$

is an element of A, B and $A \cap B$ – in all three cases with the truth value $\frac{1}{2}$.

References

1. M. Davis, *Applied nonstandard analysis*, Wiley, New York, 1977.
2. E.P. Klement, R. Mesiar, *Triangular norms*, Tatra Mountains Math. Publ. **13** (1997), 169 – 193.

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