

A NOTE ON REGULARITIES IN LATTICE GROUPS

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ABSTRACT. The goal of this note is to introduce a new property (Σ -property) of a lattice group G and to present relationships between this new concept and those ones used so far in the theory of vector lattice (lattice group) valued measures and integrals.

INTRODUCTION

Basic notions and a notation in this paper are used in the sense of [1] and [4]. For convenience of the reader we start by recalling definitions of various types of regularities in lattice groups. We use the term 'regularity' although, saying more precisely, there are simply properties and one of them is the type of distributivity.

Let G be a σ -complete lattice group and (a_{ij}) be a bounded, double sequence in G such that $a_{ij} \searrow 0$ ($j \rightarrow \infty$) for each $i \in \mathbb{N}$. In accordance with [2] in such case we will say that (a_{ij}) is a (D)-sequence.

Using the concept of (D)-sequence we recall that a σ -complete lattice group G is said to be a *g-regular* group if

$$\bigwedge \left\{ \sum_{i=1}^{\infty} a_{i\varphi(i)} \mid \varphi : \mathbb{N} \rightarrow \mathbb{N} \right\} = 0,$$

whenever (a_{ij}) is a (D)-sequence in G . Note that g-regularity implicitly requires that there exists $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ for which $\sum_{i=1}^n a_{i\varphi(i)}$ is bounded so that $\sum_{i=1}^{\infty} a_{i\varphi(i)}$ is an element of G .

The next concept of *Egoroff property* can be introduced in several equivalent ways, see [4], Theorem 67.7. G is said to have *Egoroff property* if to any (D)-sequence (a_{ij}) there exist a sequence (b_i) in G , $b_i \searrow 0$ ($i \rightarrow \infty$) and $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ such that $b_i \geq a_{i\varphi(i)}$ holds for each $i \in \mathbb{N}$.

Let us recall that a σ -complete lattice group G is said to be *weakly σ -distributive* if

$$\bigwedge \left\{ \bigvee_{i=1}^{\infty} a_{i\varphi(i)} \mid \varphi : \mathbb{N} \rightarrow \mathbb{N} \right\} = 0$$

for any (D)-sequence (a_{ij}) .

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The concept of regular Riesz space V can be introduced by several ways. Kantorovitch's conception is based on the following two properties of the space:

- (1) σ -convergence in V is stable
- (2) V has the σ -property.

In [4] (chap.10) it is proved that in σ -complete Riesz space the concept of regular Riesz space is equivalent with the following requirement (this formulation does not need the linear structure so it works in lattice groups):

If $(a_{ij}) \in V$ such that $a_{ij} \searrow 0$ ($j \rightarrow \infty$) for each $i \in \mathbb{N}$ then there exist (b_k) , $b_k \searrow 0$ ($k \rightarrow \infty$) and $\varphi : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ such that $b_k \geq a_{i\varphi(k,i)}$ holds for each $k, i \in \mathbb{N}$. (Note that the doubled (a_{ij}) is not required to be bounded from above.)

Σ -PROPERTY

In study of the relations between various 'regularities' in ordered spaces it turned out that some published results are worth to be slightly revised. We start with the note on Lemma in [7] to point out the detail which occurred to be of some importance in stating relations between the above concepts of regularities.

It is well known that in any lattice group G the following simple distributive law holds:

$$c \wedge \left(\bigvee_{n=1}^{\infty} b_n \right) = \bigvee_{n=1}^{\infty} (c \wedge b_n)$$

whenever $\bigvee b_n$ exists. The following example illustrates the situation in which a non decreasing sequence (b_n) is unbounded one so that $\bigvee b_n = \infty$ and the above equality does not hold even G is a σ -complete lattice group so that the supremum on the right side exists (i.e. is an element of G).

Example 1. Let l_∞ be the sequence space whose elements are all bounded real sequences. Let the ordering be coordinatewise. Set $c = (1, 1, 1, \dots, 1, \dots)$ and b_n are defined by $b_n(k) = k - 1$, for $k = 1, 2, \dots, n$ and $b_n(k) = n$, for $k = n + 1, n + 2, \dots$.

In this setting the left side $c \wedge (\bigvee b_n)$ of the distributive formula is equal to c if we accept convention $c \wedge \infty = c$, whereas, the right side of the formula $\bigvee (c \wedge b_n)$ is $(0, 1, 1, 1, \dots, 1, \dots)$. Apart from this if we set $d = (0, 1, 1, \dots, 1, \dots)$ then although $c \wedge b_n \leq d$ holds for every n , it does not imply $c \wedge (\bigvee b_n) \leq d$.

The next proposition is revised version of the Lemma from [7].

Proposition 1. Let G be a σ -complete lattice group and (a_{ij}) be a (D) -sequence. Then to every $c \in G$, $c > 0$ there exists a (D) -sequence (b_{ij}) such that

$$c \wedge \left(\sum_{i=1}^{\infty} a_{i\varphi(i)} \right) \leq \bigvee_{i=1}^{\infty} b_{i\varphi(i)}$$

holds for every $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ for which $\sum_{i=1}^{\infty} a_{i\varphi(i)} \in G$, i.e. for every $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ for which $(\sum_{i=1}^n a_{i\varphi(i)})$ is bounded from above.

The correction of Lemma consists in the phrase: "for every $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ for which $\sum_{i=1}^{\infty} a_{i\varphi(i)} \in G$ ". The inequality

$$c \wedge \left(\sum_{i=1}^n a_{i\varphi(i)} \right) \leq \bigvee_{i=1}^{\infty} b_{i\varphi(i)}$$

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is proved in [7] correctly but the last step from

$$c \wedge \left(\sum_{i=1}^n a_{i \varphi(i)} \right) \leq \bigvee_{i=1}^{\infty} b_{i \varphi(i)}$$

to

$$c \wedge \left(\sum_{i=1}^{\infty} a_{i \varphi(i)} \right) \leq \bigvee_{i=1}^{\infty} b_{i \varphi(i)}$$

is possible only if $(\sum_{i=1}^n a_{i \varphi(i)})$ is bounded from above. The next example illustrates that this is not always the case.

Example 2. Consider in l_{∞} (D)-sequence (a_{ij}) defined by

$$a_{ij}(k) = 0, \text{ for } k = 1, 2, \dots, j, \quad a_{ij}(k) = \frac{1}{2^i}, \text{ for } k = j+1, j+2, \dots, 2j,$$

$$a_{ij}(k) = 1, \text{ for } k = 2j+1, 2j+2, \dots, \text{ i.e.}$$

$$a_{ij} = (0, 0, \dots, 0, \frac{1}{2^i}, \frac{1}{2^i}, \dots, \frac{1}{2^i}, 1, 1, \dots, 1, \dots).$$

It is easy to see that (a_{ij}) is a (D)-sequence. Moreover, it is evident that there exist $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ for which $(\sum_{i=1}^n a_{i \varphi(i)})$ are bounded but, on the other hand, there exist $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ for which $(\sum_{i=1}^{\infty} a_{i \varphi(i)})$ do not exist in G .

We point out that in this space it can happen for (D)-sequence (a_{ij}) that there does not exist any $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ for which $(\sum_{i=1}^{\infty} a_{i \varphi(i)})$ exists in G . To illustrate this let us define (a_{ij}) by the following formula:

$$a_{ij}(k) = 0 \text{ for } k = 1, 2, \dots, j-1 \text{ and each } i \in \mathbb{N}$$

$$a_{ij}(k) = 1 \text{ for } k = j, j+1, \dots \text{ and each } i \in \mathbb{N}.$$

It can be easily verified that for every $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ the sequence $(\sum_{i=1}^n a_{i \varphi(i)})$ is not bounded from above.

Taking the above example in mind we introduce the following concept of summability of a (D)-sequence in G . We will say that (a_{ij}) is **summable** if there exists $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ for which $\sum_{i=1}^n a_{i \varphi(i)}$ is bounded (i.e. in σ -complete lattice group G the element $\sum_{i=1}^{\infty} a_{i \varphi(i)}$ exists).

Proposition 2. If G is a σ -complete and weakly σ -distributive lattice group then

$$\bigwedge \left\{ \sum_{i=1}^{\infty} a_{i \varphi(i)} \mid \varphi : \mathbb{N} \rightarrow \mathbb{N} \right\} = 0,$$

for any (D)-sequence (a_{ij}) which is summable.

Proof. Let (a_{ij}) be a summable (D)-sequence and c be a lower bound for

$$\left\{ \sum_{i=1}^{\infty} a_{i \varphi(i)} \mid \varphi : \mathbb{N} \rightarrow \mathbb{N} \right\}$$

It is evident that $d = c \vee 0$ is also the lower bound of the above set and $d \geq 0$. If $d > 0$ then according to Proposition 1 there exists (D)-sequence (b_{ij}) such that

$$d = d \wedge \left(\sum_{i=1}^{\infty} a_{i \varphi(i)} \right) \leq \bigvee_{i=1}^{\infty} b_{i \varphi(i)}$$

for all those $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ for which $\sum_{i=1}^{\infty} a_{i\varphi(i)} \in G$. For such $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ we have just obtained $d \leq \bigvee_{i=1}^{\infty} b_{i\varphi(i)}$.

Let $\varphi^{\infty} : \mathbb{N} \rightarrow \mathbb{N}$ be such that $\sum_{i=1}^{\infty} a_{i\varphi^{\infty}(i)} = \infty$. We want to show that for the above element d the inequality

$$d \leq \bigvee_{i=1}^{\infty} b_{i\varphi^{\infty}(i)}$$

holds.

According to the assumption, (a_{ij}) is summable so that there exists $\varphi^0 : \mathbb{N} \rightarrow \mathbb{N}$ such that $\sum_{i=1}^{\infty} a_{i\varphi^0(i)} \in G$. Let us define a mapping $\psi : \mathbb{N} \rightarrow \mathbb{N}$ by the formula: $\psi(i) = \max(\varphi^0(i), \varphi^{\infty}(i))$ for $i = 1, 2, 3, \dots$. The inequalities

$$\sum_{i=1}^n a_{i\psi(i)} \leq \sum_{i=1}^n a_{i\varphi^0(i)} \leq \sum_{i=1}^{\infty} a_{i\varphi^0(i)}$$

give the fact that $\sum_{i=1}^{\infty} a_{i\psi(i)} \in G$ and according to Proposition 1 we get

$$d = d \wedge \left(\sum_{i=1}^{\infty} a_{i\psi(i)} \right) \leq \bigvee_{i=1}^{\infty} b_{i\psi(i)} \leq \bigvee_{i=1}^{\infty} b_{i\varphi^{\infty}(i)}$$

Thus we have proved $d \leq \bigvee_{i=1}^{\infty} b_{i\varphi(i)}$ for any $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ what is the contradiction with the weak σ -distributivity of G . Hence $d = 0$ so that $c \leq 0$ and this means that 0 is the greatest lower bound of

$$\left\{ \sum_{i=1}^{\infty} a_{i\varphi(i)} \mid \varphi : \mathbb{N} \rightarrow \mathbb{N} \right\}.$$

This completes the proof.

With respect to the above relations let us introduce the class of spaces in which any (D)-sequence is summable.

Definition. The σ -complete lattice group G is said to have Σ -property whenever any (D)-sequence (a_{ij}) is summable, i.e. for every (D)-sequence (a_{ij}) there exists $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ such that $\sum_{i=1}^n a_{i\varphi(i)}$ is bounded (i.e. $\sum_{i=1}^{\infty} a_{i\varphi(i)} \in G$).

Proposition 3. Let G be a σ -complete lattice group having Σ -property. Then G is weakly σ -distributive iff G is g -regular.

Proof. If (a_{ij}) is a (D)-sequence then $\bigvee_{i=1}^n a_{i\varphi(i)} \leq \sum_{i=1}^n a_{i\varphi(i)}$ so that the g -regularity implies the weak σ -distributivity of G . The reverse implication is the consequence of Proposition 2.

In [6] it was proved that in any σ -complete lattice group the following chain of implications holds:

$$g\text{-regularity} \Rightarrow \text{Egoroff property} \Rightarrow \text{weak } \sigma\text{-distributivity}.$$

According to the above results in σ -complete lattice groups with Σ -property all three 'regularities' are equivalent. The last example shows that the class of Riesz spaces with Σ -property is strictly wider than the class of regular Riesz spaces.

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Example 3. Let V be a space of those real sequences that have only finite many nonzero terms. Let the ordering be pointwise. It is easy to see that this space has not σ -property so it is not regular Riesz space. On the other hand, it is not difficult to prove that it has Σ -property.

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