CENTRAL IDEALS OF GENERALIZED EFFECT ALGEBRAS AND GENERALIZED D-ALGEBRAS.

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ABSTRACT.
For generalized effect algebra (D-algebra), conditions equivalent to the condition that a nonempty subset is a central ideal are given. We show that in spite of the fact that a generalized effect algebra and the derived generalized D-algebra (or conversely) have different sets of subalgebras (in general), notions of their central ideals coincide. More precisely, they have equivalent sets of direct product decompositions in two factors (central ideals).

1. INTRODUCTION AND BASIC NOTIONS

Effect algebras (introduced by Foulis D.J. and Bennett M.K. ([1], 1994)) is important for modelling unsharp measurements in Hilbert space: The set of all effects is the set of all self adjoint operators T on a Hilbert space H with $0 \leq T \leq 1$. In a general algebraic form an effect algebra is defined in [1].

Kôpka ([6], 1992) introduced a new algebraic structure of fuzzy sets, a D-poset of fuzzy sets. A difference of comparable fuzzy sets is a primary operation in this structure. Later, Kôpka and Chovanec ([7], 1994) by transferring the properties of a difference operation of D-poset of fuzzy sets to an arbitrary partially ordered set obtained a new algebraic structure, a D-poset that generalizes orthoalgebras and MV algebras.

Generalizations of effect algebras, D-posets and D-algebras are studied by Kôpka and Chovanec (difference posets), Foulis and Bennett (Cones), Kalmbach and Riečanová (abelian RI-posets and abelian RI semigroups), Hedlíková and Pulmannová (generalized D-posets and cancellative positive partial abelian semigroups). As we can show that all above mentioned generalizations of effect algebras (cones, abelian RI-semigroups, cancellative positive PAS) are mutually equivalent algebraic structures and can be derived from generalized D-posets (deriving $\oplus$ from $\ominus$) and conversely, we will call them all generalized effect algebras. Thus their common definitions are the following:

Definition 1.1. A partial algebra $(E; \oplus, 0)$ is called a *generalized effect algebra* if $0 \in E$ is a distinguished element and $\oplus$ is a partially defined binary operation E which satisfies the following conditions for any $a, b, c \in E$:

(GEi) $a \oplus b = b \oplus a$, if one side is defined,
(GEii) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$, if one side is defined,
(GEiii) $a \oplus 0 = a$ for all $a \in E$.
(GEiv) $a \oplus b = a \oplus c$ implies $b = c$ (cancellation law),
(GEv) $a \oplus b = 0$ implies $a = b = 0$.

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Definition 1.2. A partial algebra $(P;\ominus,0)$ is called a *generalized D-algebra* (*D-poset*) if $0 \in P$ is a distinguished element $\ominus$ is a partially defined binary operation on P which satisfies the following conditions for any $a,b,c \in P$:

(GDi) $a \ominus 0 = a$ for all $a \in P$,
(GDii) $a \ominus a = 0$ for all $a \in P$,
(GDiii) if $b \ominus a$ is defined then $b \ominus (b \ominus a)$ is defined,
(GDiv) $(a \ominus b) \ominus c = (a \ominus c) \ominus b$, if one side is defined,
(GDv) if $b \ominus a$ and $c \ominus b$ are defined then $c \ominus a$ is defined,
(GDvi) if $a \ominus b$ and $b \ominus a$ are defined then $a = b$
(GDvii) $c \ominus a = b \ominus a$ implies $c = b$ (cancellation law).

Proposition 1.3. (1) *In every generalized D-algebra $(P;\ominus,0)$ the partial binary operation $\oplus$ can be defined by*

(GED) *$a \oplus b$ is defined and $a \oplus b = c$ iff $c \ominus a$ is defined and $c \ominus a = b$ and the partial order in P can be defined by*
$a \leq b$ iff $b \ominus a$ is defined.

(2) *In every generalized effect algebra $(E;\oplus,0)$ the partial binary operation $\ominus$ can be defined by*

(GDE) *$a \ominus b$ is defined and $a \ominus b = c$ iff $b \oplus c$ is defined and $b \oplus c = a$ and the partial order in P can be defined by*
$a \leq b$ iff there exists $c \in P$ with $a \oplus c = b$.

Cancellation laws (GEiv) and (GDvii) guarantee that $\oplus$, $\ominus$ and $\leq$ are well defined. Moreover we can show the following statements:

Proposition 1.4. (1) *In every generalized effect algebra $(E;\oplus,0)$ the partial binary operation $\ominus$ derived by* (GDE) *fulfills axioms* (GDi)–(GDvii) *of a generalized D-algebra.*

(2) *In every generalized D-algebra $(P;\ominus,0)$ the partial binary operation $\oplus$ derived by* (GED) *fulfills axioms of a generalized effect algebra.*

Moreover, the partial orders derived from the corresponding operations $\oplus$ and $\ominus$, i.e. derived one from the other by (GED) *respectively* (GDE) *coincide.*

Remark 1.5. In view of Proposition 1.4 we may consider both a generalized effect algebra $(P;\oplus,0)$ and the derived generalized D-algebra $(P;\ominus,0)$ (and conversely) as a set P with $0, \leq, \ominus$ and $\oplus$ satisfying all properties (GEi)–(GEv), (GDi)–(GDvii) and (GED) and (GDE). On the other hand a generalized D-algebra with a fundamental operation $\ominus$ and derived $\oplus$ and the (derived) generalized effect algebra with the fundamental operation $\oplus$ and derived $\ominus$ are different from some algebraic points of view, e.g. they have different sets of subalgebras (see [9]).

The following proposition follows from Proposition 1.3.

Proposition 1.6. *Let $(P;\oplus,0)$ and $(P;\ominus,0)$ be a generalized effect algebra and a generalized D-algebra derived one from the other. Let $\emptyset \neq Q \subseteq P$. Then $(Q;\oplus,0)$ and $(Q;\ominus,0)$, with inherited operations are a generalized effect algebra and a generalized D-algebra in their own right derived one from the other and partial order in Q is inherited from P if and only if the following condition is satisfied*

(S) *If from elements $a,b,c \in P$ with $a \oplus b = c$ (or equivalently $c \ominus b = a$) at least two are elements of Q then $a,b,c \in Q$.*

Definition 1.7. For a generalized effect algebra $(P;\oplus,0)$ a set $\emptyset\neq Q\subseteq P$ satisfying condition (S) is called a *sub-generalized effect algebra* denoted by $(Q;\oplus,0)$.

Definition 1.8. For a generalized effect algebra $(P;\ominus,0)$ a set $\emptyset\neq Q\subseteq P$ satisfying condition (S) is called a *sub-generalized D-algebra* denoted by $(Q;\ominus,0)$.

Definition 1.9. For a generalized effect algebra $(P;\oplus,0)$ (resp. a generalized D-algebra $(P;\ominus,0)$) a set $\emptyset\neq Q\subseteq P$ is called an *ideal* if Q has property (S) and for all $b\in Q, a\in P$: $a\leq b$ implies $a\in Q$.

Definition 1.10. Let $(P;\oplus_P,0_P)$ and $(Q;\oplus_Q,0_Q)$ are generalized effect algebras. A map $\varphi: P\to Q$ is called an *isomorphism* if φ is bijective and for all $a,b\in P$, $a\oplus_P b$ is defined iff $\varphi(a)\oplus_Q\varphi(b)$ is defined, in which case $\varphi(a\oplus_P b)=\varphi(a)\oplus_Q\varphi(b)$.

Lemma 1.11. *Let $(P;\oplus_P,0_P)$ and $(Q;\oplus_Q,0_Q)$ be generalized effect algebras with partial orders $\leq_P$ on P and $\leq_Q$ on Q derived from $\oplus_P$ and $\oplus_Q$ respectively. Let $\varphi: P\to Q$ be an isomorphism. Then for $a,b\in P$*
(i) *$a\leq_P b$ iff $\varphi(a)\leq_Q\varphi(b)$,*
(ii) *$a\vee_P b$ exists iff $\varphi(a)\vee_Q\varphi(b)$ exists in which case $\varphi(a\vee_P b)=\varphi(a)\vee_Q\varphi(b)$,*
(iii) *$a\wedge_P b$ exists iff $\varphi(a)\wedge_Q\varphi(b)$ exists in which case $\varphi(a\wedge_P b)=\varphi(a)\wedge_Q\varphi(b)$*

The proof is left to the reader.

2. A DIRECT PRODUCT DECOMPOSITION OF GENERALIZED EFFECT ALGEBRAS

Definition 2.1. A generalized effect algebra $(P;\oplus,0)$ is a *direct product of generalized effect algebras* $(P_1;\oplus_1,0_1)$ and $(P_2;\oplus_2,0_2)$ if $P=P_1\times P_2$ (i. e. the set P is the cartesian product of sets P_1 and P_2), $0=(0_1,0_2)$ and for $(x_1,y_1),(x_2,y_2)\in P$, $(x_1,y_1)\oplus(x_2,y_2)$ is defined iff $x_1\oplus_1 x_2$, $y_1\oplus_2 y_2$ are defined in which case $(x_1,y_1)\oplus(x_2,y_2)=(x_1\oplus_1 x_2, y_1\oplus_2 y_2)$.

Definition 2.2. Let $(P;\oplus,0)$ be a generalized effect algebra. If $P_1,P_2\subseteq P$ are sub-generalized effect algebras of P such that $(P;\oplus,0)$ is isomorphic to a direct product of $(P_1;\oplus,0)$ and $(P_2;\oplus,0)$ (written $P\cong P_1\times P_2$) then we say that a generalized effect algebra $(P;\oplus,0)$ has a *direct product decomposition* $P_1\times P_2$. The direct product decomposition $P_1\times P_2$ of $(P;\oplus,0)$ is *trivial* if one of P_1 and P_2 equals to P. A generalized effect algebra $(P;\oplus,0)$ is called *irreducible* if it has only trivial direct product decompositions.

Note that if a generalized effect algebra $(P;\oplus,0)$ has a direct product decomposition $P_1\times P_2\cong P$ for some sub-generalized effect algebras $(P_1;\oplus_{P_1},0_{P_1})$ and $(P_2;\oplus_{P_2},0_{P_2})$ then $\tilde{P}_1=\{(p,0)\mid p\in P_1\}$ and $\tilde{P}_2=\{(0,q)\mid q\in P_2\}$ are sub-generalized effect algebras of $P_1\times P_2$. This follows by the property $a\oplus b=0$ iff $a=b=0$, for all $a,b\in P$. Actually we have $P_1\cong\tilde{P}_1$ and $P_2\cong\tilde{P}_2$. The last imply that *for every $p\in P_1$ and every $q\in P_2$ there exist $p\oplus q=p\vee q$ and $p\wedge q=0$.*

Definition 2.3. For a subset $\emptyset\neq M\subseteq P$ of a generalized effect algebra $(P;\oplus,0)$ we denote $M^\perp=\{x\in P\mid \text{for every } y\in M \text{ there exist } x\oplus y=x\vee y\}$

Definition 2.4. A subset $\emptyset\neq Z\subseteq P$ of a generalized effect algebra $(P;\oplus,0)$ is called a *central ideal* if $(P;\oplus,0)$ has a direct product decomposition $Z\times Z^\perp$ (written $P\cong Z\times Z^\perp$).

Theorem 2.5. *Let $(P;\oplus,0)$ be a generalized effect algebra and let $\emptyset \neq Z \subseteq P$ be such that $P = Z \vee Z^{\perp} = \{p \vee q \mid p \in Z, q \in Z^{\perp}\}$. Then*

(i) *for every $x \in P$ there exist unique $p \in Z$ and $q \in Z^{\perp}$ such that $x = p \vee q = p \oplus q$,*

(ii) *Z and $Z^{\perp}$ are order ideals in $(P;\leq)$, where $\leq$ is derived from $\oplus$.*

(iii) *Z and $Z^{\perp}$ are ideal subalgebras of $(P;\oplus,0)$*

(iv) *$P \cong Z \times Z^{\perp}$*

Proof. (i) Let $x \in P$ and let $p, u \in Z$, $q, v \in Z^{\perp}$ be such that $x = p \vee q = u \vee v$. By definition of $Z^{\perp}$, $p \oplus q = p \vee q = x$ and $u \oplus v = u \vee v = x$. Moreover, by definition of $Z^{\perp}$ also $q \oplus u$ and $v \oplus p$ are defined. Since $u \vee q, p \vee v \in Z \vee Z^{\perp} = P$ we obtain $u \oplus q = u \vee q \leq x = p \oplus q$ and $p \oplus v = p \vee v \leq x = p \oplus q$. It follows that $u \leq p$ and $v \leq q$. Similarly, $u \oplus q = u \vee q \leq x = u \oplus v$ and $p \oplus v = p \vee v \leq x = u \oplus v$ which imply $q \leq v$ and $p \leq u$. We conclude that $u = p$ and $v = q$.

(ii) Suppose that $x \in Z$ and $y \in P$ with $y \leq x$. Then $y = p \vee q \leq x$ for some $p \in Z$ and $q \in Z^{\perp}$. It follows that $q = q \wedge x = 0$ and hence $y = p \in Z$. If $y \leq x \in Z^{\perp}$ then $y = p \vee q \leq x$ for some $p \in Z$ and $q \in Z^{\perp}$ imply that $p = p \wedge x = 0$ and $y = q \in Z^{\perp}$.

(iii) Let $x, y \in Z$ with defined $x \oplus y$. By (i) there exist $p \in Z$ and $q \in Z^{\perp}$ such that $x \oplus y = p \vee q = p \oplus q$. By the assumption $P = Z \vee Z^{\perp}$ and by definition of $Z^{\perp}$ there exist $x \vee q$ and $x \oplus q$ and $x \oplus q = x \vee q \leq x \oplus y$ which implies that $q \leq y$. It follows that $y \in Z \cap Z^{\perp} = \{0\}$. Hence $x \oplus y = p \in Z$.

Similarly, for $x, y \in Z^{\perp}$ with defined $x \oplus y$ there exist $p \in Z$ and $q \in Z^{\perp}$ with $x \oplus y = p \vee q = p \oplus q$ and $x \oplus p = x \vee p \leq x \oplus y$. It follows that $p \leq y \in Z^{\perp}$. We conclude that $p \in Z \cap Z^{\perp} = \{0\}$ and hence $x \oplus y = q \in Z^{\perp}$.

(iv) In view of (iii), Z and $Z^{\perp}$ are sub-generalized effect algebras of $(P;\oplus,0)$. Let a map $\varphi : P \to Z \times Z^{\perp}$ be defined as follows:

for every $x \in P$, $\varphi(x) = (p,q) \in Z \times Z^{\perp}$, where (by condition (i)) $p \in Z$ and $q \in Z^{\perp}$ are the unique elements for which $x = p \vee q = p \oplus q$. In view of the condition $P = Z \vee Z^{\perp}$ we obtain that φ is bijective. Evidently $\varphi(0) = (0,0)$. Further, let $x, y \in P$, where $x = p_1 \oplus q_1$, $y = p_2 \oplus q_2$ for some $p_1, p_2 \in Z$, $q_1, q_2 \in Z^{\perp}$. We obtain that $x \oplus y$ exists iff there exists $(p_1 \oplus q_1) \oplus (p_2 \oplus q_2) = (p_1 \oplus p_2) \oplus (q_1 \oplus q_2)$ and it is iff $p_1 \oplus p_2$ and $q_1 \oplus q_2$ exist, since $p_1 \oplus p_2 \in Z$ and $q_1 \oplus q_2 \in Z^{\perp}$ by (iii). Moreover $\varphi(x) \oplus \varphi(y) = (p_1, q_1) \oplus (p_2, q_2)$ exists iff $p_1 \oplus p_2$ and $q_1 \oplus q_2$ exist and that is iff $x \oplus y$ exists. Moreover, $\varphi(x \oplus y) = \varphi(x) \oplus \varphi(y)$.

Theorem 2.6. *Let a generalized effect algebra $(P;\oplus,0)$ have a direct product decomposition $Z \times Z^{\perp}$ for some $\emptyset \neq Z \subseteq P$ (i. e. $P \cong Z \times Z^{\perp}$). Then $P = Z \vee Z^{\perp} = \{p \vee q \mid p \in Z, q \in Z^{\perp}\}$.*

Proof. Let $\varphi : P \to Z \times Z^{\perp}$ be the isomorphism. For every $x \in P$ there exist unique $p \in Z, q \in Z^{\perp}$ such that $\varphi(x) = (p,q) \in Z \times Z^{\perp}$ and evidently $\varphi(p) = (p,0) \in \varphi(Z) = \{(p,0) \mid p \in Z\}$ and $\varphi(q) = (0,q) \in \varphi(Z^{\perp}) = \{(0,q) \mid q \in Z^{\perp}\}$. It follows that $x = \varphi^{-1}(\varphi(p) \oplus_{Z \times Z^{\perp}} \varphi(q)) = p \oplus q = p \vee q$. Conversely, for every $p \in Z$ and $q \in Z^{\perp}$ we have $(p,q) \in Z \times Z^{\perp}$ and hence there exists $x = \varphi^{-1}((p,q)) \in P$. Moreover, $\varphi^{-1}((p,q)) = p \vee q$.

The following result is a consequence of precedent theorems.

Theorem 2.7. *Let $(P;\oplus,0)$ be a generalized effect algebra and $\emptyset \neq Z \subseteq P$. The following conditions are equivalent:*
(i) *Z is a central ideal of $(P;\oplus,0)$,*
(ii) *$P = Z \vee Z^{\perp} = \{p \vee q \mid p \in Z, q \in Z^{\perp}\}$,*
(iii) *for every $x \in P$ there exist unique $p \in Z$ and $q \in Z^{\perp}$ such that $x = p \vee q = p \oplus q$.*

3. A DIRECT PRODUCT DECOMPOSITION OF GENERALIZED $\mathcal{D}$ - ALGEBRAS.

Definition 3.1. A generalized D-algebra $(P;\ominus,0)$ is a direct product of generalized D-algebras $(P_1;\ominus_1,0_1)$ and $(P_2;\ominus_2,0_2)$ if $P = P_1 \times P_2$, $0 = (0_1,0_2)$ and for $(x_1,y_y),(x_2,y_2) \in P$ there exists $(x_2,y_2)\ominus(x_1,y_1)$ iff there exist $y_2 \ominus_2 y_1$, $x_2 \ominus_1 x_1$, in which case $(x_2,y_2)\ominus(x_1,y_1) = (x_2 \ominus_1 x_1, y_2 \ominus_2 y_1)$.

Note that if a generalized D-algebra $(P;\ominus,0)$ has a direct product decomposition $P_1 \times P_2 \cong P$ for some sub-generalized D-algebras $(P_1;\ominus_{P_1},0_{P_1}$ and $(P_2;\ominus_{P_2},0_{P_2}$ then $\tilde{P}_1 = \{(p,0) \mid p \in P_1\}$ and $\tilde{P}_2 = \{0,q \mid q \in P_2\}$ have property (S) hence they are sub-generalized D-algebras of $P_1 \times P_2$. Actually we have $P_1 \cong \tilde{P}_1$ and $Q_1 \cong \tilde{Q}_2$. The last imply that for every $p \in P_1$ and $q \in P_2$ there exist $p \vee q$ and $(p \vee q)\ominus p = q$.

Definition 3.2. For a generalized effect algebra $(P;\ominus,0)$ and a set $\emptyset \neq M \subseteq P$ let us denote $M^{\perp} = \{q \in P \mid \text{for every } p \in M \text{ there exists } p \vee q \text{ and } (p \vee q)\ominus p = q\}$.

Obviously, if $(P;\oplus,0)$ is a generalized effect algebra derived from $(P;\ominus,0)$ then Definition 2.4 of $M^{\perp}$ coincide with Definition 3.2.

Definition 3.3. Let $(P;\ominus_P,0_P)$ and $(Q;\ominus_Q,0_Q)$ be generalized effect algebras. A map $\varphi : P \to Q$ is called an *isomorphism* if φ is bijective and for all $a,b \in P$, $b \ominus_P a$ is defined iff $\varphi(b)\ominus_Q \varphi(a)$ is defined in which case $\varphi(b \ominus_P a) = \varphi(b)\ominus_Q \varphi(a)$.

Theorem 3.4. *Let $(P;\oplus,0)$ be a generalized effect algebra and $(P;\ominus,0)$ be a generalized D-algebra derived one from the other by* (GDE). *For any $\emptyset \neq Z \subseteq P$ the following conditions are equivalent:*
(i) *$P = Z \vee Z^{\perp} = \{p \vee q \mid p \in Z, q \in Z^{\perp}\}$.*
(ii) *$(P;\oplus,0)$ is isomorphic to a direct product of sub-generalized effect algebras $(Z;\oplus,0)$ and $(Z^{\perp};\oplus,0)$.*
(iii) *$(P;\ominus,0)$ is isomorphic to a direct product of sub-generalized D-algebras $(Z;\ominus,0)$ and $(Z^{\perp};\ominus,0)$.*
(iv) *For every $x \in P$ there exists a unique $p \in Z$ such that $x \ominus p \in Z^{\perp}$ (in such case $x = p \vee (x \ominus p)$).*

Proof. By Theorem 2.5, (i) $\Rightarrow$ (ii) and (iv). By Theorem 2.6, (ii) $\Rightarrow$ (i). Evidently, (iv) $\Rightarrow$ (i).

(ii) $\Leftrightarrow$ (iii): If $(P;\oplus,0)$ is isomorphic to a direct product of sub-generalized effect algebras $(Z;\oplus,0)$, $(Z^{\perp};\oplus,0)$ then Z and $Z^{\perp}$ satisfy condition (S). By Proposition 1.6 $(Z;\ominus,0)$ and $(Z^{\perp};\ominus,0)$ are sub-generalized D-algebras of $(P;\ominus,0)$. We actually have that $(P;\ominus,0)$ is isomorphic to the direct product of $(Z;\ominus,0)$ and $(Z^{\perp};\ominus,0)$. Conversely, if $(P;\ominus,0)$ is isomorphic to a direct product of sub-generalized effect algebras $(Z;\ominus,0)$ and $(Z^{\perp};\ominus,0)$ then Z and $Z^{\perp}$ satisfy condition (S) and hence $(Z;\oplus,0)$ and $(Z^{\perp};\oplus,0)$ are sub-generalized effect algebras of $(P;\oplus,0)$. Actually, $(P;\oplus,0)$ is isomorphic to a direct product of $(Z;\oplus,0)$ and $(Z^{\perp};\oplus,0)$.

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