ON THE JOINT OBSERVABLE AND THE JOINT DISTRIBUTION IN PRODUCT MV ALGEBRAS

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ABSTRACT. A notion of a product MV algebra is presented and two existence theorems (for joint observable and joint distribution) are formulated in product MV algebras.

1. MV algebra of fuzzy sets

A prototype of MV algebra is the set $\mathcal{F}$ of all fuzzy sets $f\colon X \to \langle 0,1\rangle$ measurable with respect to a σ-algebra $\mathcal{S}$ of subsets of X. Certainly, $\mathcal{F}$ is closed with respect to many operations. The MV algebra operations are the following: two binary operations $\oplus, \odot$, where

$$f \oplus g = \min(f+g, 1),$$
$$f \odot g = \max(f+g-1, 0),$$

one unary operation $*$ with

$$f^* = 1 - f$$

and two nulary operations (fixed elements) $0_X, 1_X$. Recall that $\oplus$ can be interpreted as the composition of two pictures, if $f\colon X \to \langle 0,1\rangle$ is interpreted as a picture (0 is white colour, 1 is black colour, $\alpha \in (0,1)$ means something grey, the composition of two grey colours can not be greater than 1) and $\odot$ can be obtained by the de Morgan rule: $f \odot g = (f^* \oplus g^*)^*$. If $f = \chi_A$, $g = \chi_B$ are characteristic functions, then $f \oplus g = \chi_{A \cup B}$, $f \oplus g = \chi_{A \cap B}$, $f^* = \chi_A$.

Generally MV algebra is an algebraic system $(M, \oplus, \odot, *, 0, u)$ satisfying some properties. Of course, following the Mundici representation theorem it is more convenient to define MV algebra by the help of lattice ordered groups.

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2. Commutative lattice ordered groups

Commutative lattice ordered group is an algebraic system $(G, +, \leq)$ with the following properties:

1. $(G, +)$ is a commutative group.
2. $(G, , \leq)$ is a partially ordered set being a lattice, i.e., to every $a, b \in G$ there exists the least upper bound $a \vee b$ and the greatest lower bound $a \wedge b$.
3. If $a, b, c \in G$ and $a \leq b$, then $a + c \leq b + c$.

These axioms have many useful consequences, as $a + (b \vee c) = (a + b) \vee (a + c)$, $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ etc.

A typical example of a commutative l-group is the set of all real-valued functions defined on a set. Here $f + g$ is the usual sum of two functions and $f \leq g$ if and only if $f(x) \leq g(x)$ for any $x \in X$.

Another example of a commutative ℓ-group is the set of all functions $f \colon X \to R$ measurable with respect to a σ-algebra of subsets of X. This example is related to the first example presented in Section 1. This is a key to the notion of general MV algebra. It is sufficient to define in a commutative ℓ-group MV algebra operations similarly as it was done in the mentioned example.

3. MV algebra

Let $(G, +, \leq)$ be a commutative lattice ordered group and $u \in G$ be any element such that $u > 0$, i.e., $u \geq 0$ and $u \neq 0$. Put

$$M = \langle 0, u \rangle = \{v \in G;\ 0 \leq v \leq u\}.$$

Define further (analogously to the example presented in Section 1) two binary operations $\oplus, \odot$ by the formulas

$$a \oplus b = (a + b) \wedge u\,,$$
$$a \odot b = (a + b - u) \vee 0$$

and a unary operation $*$ by the formula

$$a^* = u - a\,.$$

Then the algebraic system $(M \oplus, \odot, *, 0, u)$ is called an MV algebra.

If we consider the ℓ-group $(R^X, +, \leq)$ of all real-valued functions on a set X, then its subset $M = \{f \colon X \to R;\ 0 \leq f \leq 1\}$ is an MV algebra

$(M, \oplus, \odot, *, 0_X, 1_X)$, where

$$f \oplus g = (f+g) \wedge 1_X\,,$$
$$f \odot g = (f+g-1_X) \vee 0_X\,,$$
$$f^* = 1_X - f\,.$$

Remark that $a \oplus b = a+b$, if $a \leq b^*$. Indeed, $a \leq b^* = u-b$ implies $a+b \leq u$, hence $a \oplus b = (a+b) \wedge u = a+b$.

4. States and observables

The notion of a state corresponds to the notion of a probability measure in the Kolmogorov model, the notion of an observable corresponds to the notion of a random variable.

Recall that a probability measure is a normed, additive and continuous set-function defined on a σ-algebra. If we substitute the σ-algebra by an arbitrary MV algebra, we obtain the following definition.

DEFINITION. A state m on an MV algebra $M = (M, \oplus, \odot, *, 0, u)$ is a mapping $m\colon M \to \langle 0,1\rangle$ satisfying the following properties:

(i) $m(u) = 1$

(ii) If $a, b, c \in M$ and $a = b + c$, then $m(a) = m(b) + m(c)$.

(iii) If $a_n \in M$ $(n = 1, 2, \dots)$, $a \in M$ and $a_n \nearrow a$, then $m(a_n) \nearrow m(a)$.

Recall that a random variable is an $\mathcal{S}$-measurable function $\xi\colon X \to R$ defined on a measurable space $(X, \mathcal{S})$, where $\mathcal{S}$ is a σ-algebra, i.e., $\xi^{-1}(B) \in \mathcal{S}$ for any Borel set $B \in \mathcal{B}(R)$. If we assigne to any $B \in \mathcal{B}(R)$ its preimage $\xi^{-1}(B) \in \mathcal{S}$, then we obtain a σ-morphism from $\mathcal{B}(R)$ to $\mathcal{S}$. Therefore it is natural to consider an observable in our MV algebra model as a morphism $x\colon \mathcal{B}(R) \to M$.

DEFINITION. A weak observable (with respect to a state m) is a mapping $x\colon \mathcal{B}(R) \to M$ satisfying the following conditions:

(i) $m(x(R)) = 1$.

(ii) If $A \cap B = \emptyset$, then $x(A \cup B) = x(A) + x(B)$.

(iii) If $A_n \nearrow A$, then $x(A_n) \nearrow x(A)$.

A weak observable is called observable, if $x(R) = u$.

PROPOSITION. *If $m\colon M \to \langle 0,1\rangle$ is a state and $x\colon \mathcal{B}(R) \to M$ is an observable, then $m_x = m \circ x\colon \mathcal{B}(R) \to \langle 0,1\rangle$ is a probability measure.*

5. Joint observable

The notion of a joint observable in our model corresponds to the notion of a random vector in the Kolmogorov model. Recall that a random vector T is a couple of random variables, $T = (\xi, \eta)$, hence

$$T = (\xi, \eta) \colon X \to R^2 .$$

If we assigne to any Borel set $B \in \mathcal{B}(R^2)$ its preimage $T^{-1}(B) \in \mathcal{S}$, we obtain a morphism

$$\mathcal{B}(R^2) \to \mathcal{S}, \quad B \mapsto T^{-1}(B) .$$

Moreover

$$T^{-1}(C \times D) = \xi^{-1}(C) \cap \eta^{-1}(D) \tag{$*$}$$

for any $C, D \in \mathcal{B}(R)$.

Let us return now to the MV algebra $\mathcal{F}$ of fuzzy sets (Section 1) consisting of all measurable functions $f \colon X \to \langle 0, 1\rangle$. Consider two observables $x, y \colon \mathcal{B}(R) \to \mathcal{F}$. We want to define the joint observable of x, y. It should be a morphism

$$h \colon \mathcal{B}(R^2) \to \mathcal{F}$$

satisfying some condition analogous to $(*)$. Of course, instead of intersection of the sets $\xi^{-1}(C)$ and $\eta^{-1}D)$ we need to consider the intersection of fuzzy sets $x(C)$ and $y(D)$, where $x(C)$ and $y(D)$ are functions from X to $\langle 0, 1\rangle$. Of course, there exists infinitely many possibilities how to define the intersection of fuzzy sets. But the only one is suitable for us: the usual product $x(C) \cdot y(D)$ of two real functions $x(C), y(D)$. Namely only in this case the couple of operations $+, \cdot$ fulfills the distributive law.

DEFINITION. The joint observable of two weak observables $x, y \colon \mathcal{B}(R) \to \mathcal{F}$ is a mapping $h \colon \mathcal{B}(R^2) \to \mathcal{F}$ satisfying the following conditions:

(i) $m(h(R^2)) = 1$.

(ii) If $A \cap B = \emptyset$, then $h(A \cup B) = h(A) + h(B)$.

(iii) If $A_n \nearrow A$, then $h(A_n) \nearrow h(A)$.

(iv) If $C, D \in \mathcal{B}(R)$, then $h(C \times D) = x(C) \cdot y(D)$.

It is not difficult to prove ([4]) that the joint observable exists for any observables $x, y \colon \mathcal{B}(R) \to \mathcal{F}$.

6. Product MV algebra

We want to define the joint observable in a general MV algebra M. Of course, it is necessary to have a product of two elements. Therefore, we shall assume that there is given a binary operation $\cdot$ on M satisfying some axioms.

DEFINITION. A product MV algebra is an MV algebra $M = (M, \oplus, \odot, *, 0, u)$ together with a binary operation $\cdot$ on M satisfying the following conditions:

(i) $u \cdot u = u$.

(ii) The operation $\cdot$ is associative.

(iii) If $a + b \leq u$, then $c \cdot (a+b) = c \cdot a + c \cdot b$ and $(a+b) \cdot c = a \cdot c + b \cdot c$ for any $c \in M$.

(iv) If $a_n \nearrow 0$, $b_n \searrow 0$, then $a_n \cdot b_n \searrow 0$.

Evidently, the MV algebra $\mathcal{F}$ of fuzzy sets (Section 1) is a product MV algebra. Now the notion of a joint observable can be introduced in any product MV algebra.

DEFINITION. Let M be a product MV algebra, $x, y \colon \mathcal{B}(R) \to M$ be weak observables. The joint observable of x, y is a mapping $h \colon \mathcal{B}(R^2) \to M$ satisfying the following conditions:

(i) $m\big(h(R^2)\big) = 1$.

(ii) If $A \cap B = \emptyset$, then $h(A \cup B) = h(A) + h(B)$.

(iii) If $A_n \nearrow A$, then $h(A_n) \nearrow h(A)$.

(iv) If $C, D \in \mathcal{B}(R)$, then $h(C \times D) = x(C) \cdot y(D)$.

7. The joint observable extension theorem

We are not able to prove the existence of the joint observable in any MV algebra. Therefore we shall restrict our considerations to so-called weakly σ-distributions MV algebras.

DEFINITION. An MV algebra M is σ-complete, if any sequence (x_n) of elements of M has in M the least upper bound $\bigvee\limits_n x_n$. A σ-complete MV algebra is weakly σ-distributive, if for any bounded sequence $(a_{ij})_{i,j}$ such that $a_{ij} \searrow 0$ $(j \to \infty,\ i = 1, 2, \dots)$ it is

$$\bigwedge_{\varphi \in N^N} \bigvee_{i=1}^{\infty} a_{i\varphi(i)} = 0 .$$

Weak σ-distributivity is really a kind of distributivity. Namely, if $\bigwedge\limits_j a_{ij} = 0$ by the assumption, then

$$\bigwedge_{\varphi} \bigvee_{i} a_{i\varphi(i)} = \bigvee_{i} \bigwedge_{j} a_{ij} = 0 .$$

Another view is given by the real case. If (a_{ij}) is a bounded double sequence of real numbers such that $a_{ij} \downarrow 0$ $(j \to \infty)$, then to every $\varepsilon > 0$ and every $i \in N$ there exists $\varphi(i) \in N$ such $a_{ij} < \varepsilon$ for any $j \geq \varphi(i)$. Particularly

$$a_{i\varphi(i)} < \varepsilon ,$$

hence

$$\bigvee_{i=1}^{\infty} a_{i\varphi(i)} \leq \varepsilon . \tag{+}$$

Since to every $\varepsilon > 0$ there exists $\varphi \colon N \to N$ such that (+) holds, we obtain

$$\bigwedge_{\varphi \in N^N} \bigvee_{i=1}^{\infty} a_{i\varphi(i)} = 0 .$$

Recall that the weak σ-distributivity is a necessary conditions for a Riesz space G for any G-valued measure could be extended from a ring to the generated σ-ring ([11]).

The first of two main results presented in the paper is the following:

THEOREM 1. ([8]). *Let M be a weakly σ-distributive product MV algebra. Then to any observables $x, y \colon \mathcal{B}(R) \to M$ there exists their joint observable.*

8. The joint distribution existence theorem

If x, y are two observables and h their joint observable, we can construct the composit mapping

$$m_h = m \circ h \colon \mathcal{B}(R^2) \to \langle 0, 1 \rangle .$$

It is easy to see that m_h is a probability measure such that

$$m_h(C \times D) = m\big(h(C \times D)\big) = m\big(x(C) \cdot y(D)\big)$$

for any $C, D \in \mathcal{B}(R)$.

In the Kolmogorov model, m_h is the probability distribution corresponding to given random variables. Indeed, if $\xi, \eta \colon X \to R$ are random variables and $T = (\xi, \eta)$ is the corresponding random vector, then its probability distribution $P_T \colon \mathcal{B}(R^2) \to \langle 0, 1 \rangle$ is defined by

$$P_T(B) = P\big(T^{-1}(B)\big) ,$$

hence

$$P_T(C \times D) = P\big(T^{-1}(C \times D)\big) = P\big(\xi^{-1}(C) \cap \eta^{-1}(D)\big) .$$

DEFINITION. We say that a probability measure $\mu\colon \mathcal{B}(R^2) \to \langle 0,1\rangle$ is the joint distribution of observables $x, y\colon \mathcal{B}(R) \to M$, if

$$\mu(C \times D) = m\big(x(C) \cdot y(D)\big)$$

for any $C, D \in \mathcal{B}(R)$.

THEOREM 2. ([9]). *For any product MV algebra and any observables $x, y\colon \mathcal{B}(R) \to M$ there exists their joint distribution.*

9. Applications

The first important application of the joint observable is a possibility to built a calculus with observables. If $h\colon \mathcal{B}(R^2) \to M$ is the joint observable of observables $x, y\colon \mathcal{B}(R) \to M$, then the sum $x + y\colon \mathcal{B}(R) \to M$ can be defined by the formula

$$x + y = h \circ g^{-1},$$

where $g\colon R^2 \to R$ is defined by $g(u,v) = u + v$. This formula can be justified by the classical case. If $T = (\xi, \eta)$ is a random vector, then

$$\xi + \eta = g \circ T,$$

hence

$$(\xi + \eta)^{-1} = T^{-1} \circ g^{-1}.$$

The second important application of the notion of joint observable is the following. Consider a sequence $(y_n)_n$ of observables. Let $h_n\colon \mathcal{B}(R^n) \to M$ be the joint observable of $y_1, \dots y_n$. Then $\mu_n = m \circ h_n\colon \mathcal{B}(R^n) \to \langle 0,1\rangle$ is a probability measure. To the sequence $(\mu_n)_n$ the Kolmogorov consistency theorem can be applied and a local representation of observables by random variables can be obtained ([10]). By this apparatus some probability assertions can be proved (laws of large numbers, central limit theorem etc.). As it was noted in [9] , in some problems (e.g., the definition of the conditional probability) instead of the joint observable only the joint distribution can be used.

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