

A NOTE TO INDEPENDENT EVENTS ON QUANTUM LOGICS

OLGA NÁNÁSIOVÁ

Slovak University of Technology

ABSTRACT. The classical probabilistic independence of random events can be defined in several equivalent ways. Let (Ω, \mathcal{S}, P) be a probability space, $A, B \in \mathcal{S}$, $P(A) \neq 0$, $P(B) \neq 0$. Then A and B are independent iff (1) $P(A \cap B) = P(A)P(B)$, (2) $P(A|B) = P(A|\Omega)$, (3) $P(B|A) = P(B|\Omega)$. However, the above equivalences may fail on the quantum logics. Namely, it is possible that $p(a|b) = p(a|1)$ but $p(b|a) \neq p(b|1)$.

CONDITIONAL STATES ON A QUANTUM LOGIC

A set L will be called a quantum logic (an orthomodular lattice) [4]

- (i) $L \neq \emptyset$, partially ordered set with 0 and 1 .
- (ii) For any $a, b \in L$ $a \vee b \in L$.
- (iii) There is a map $\perp : L \rightarrow L$:
 - (a) For any $a \in L$ $(a^\perp)^\perp = a$.
 - (b) If $a \in L$ then $a \vee a^\perp = 1$.
 - (c) If $a, b \in L$ such that $a \leq b$ then $b^\perp \leq a^\perp$.
 - (iv) If $a, b \in L$ such that $a \leq b$ then $b = a \vee (a^\perp \wedge b)$ (orthomodular law).

Elements $a, b \in L$ will be called: orthogonal $a \perp b$ iff $a \leq b^\perp$; compatible $a \leftrightarrow b$ iff there exist $a_1, b_1, c \in L$ mutually orthogonal, such that $a = a_1 \vee c$ and $b = b_1 \vee c$.

A map $m : L \rightarrow R$ such that

- (i) $m(0) = 0$ and $m(1) = 1$.
- (ii) If $a \perp b$ then $m(a \vee b) = m(a) + m(b)$

is called a state on L . If we have orthomodular σ -lattice and m is σ -additive function then m will be called a σ -state.

Let M be a set of states on L . The pair (L, M) will be called a quite full system (briefly q.f.s.) if $\{m \in M; m(a) = 1\} \subset \{m \in M; m(b) = 1\}$ implies $a \leq b$.

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On the quantum logic a conditional state (conditional probabilities) has been studied by several authors ([5], [6], [7], [8], [9], [10], [17]). These approaches were inspired by papers [1], [2], [3], [15], where this problem was studied on classical probability space.

Cassinelli and Beltrametti [6] have defined a transformation $\Omega_a: M_a \rightarrow M_a$ such that

$$(I.) s(\Omega_a m) = s(m) \star a, \text{ where } c \star b = (c \vee b^\perp) \wedge b.$$

Cassinelli and Truini [9] added the following properties:

(II.) Let $m \in M$ and put $L(m) = \{b \in L; m(b) > 0\}$. Then $\Omega_{(\cdot)}: L(m) \rightarrow M$ is a map, and for any $b \in L_0$ $\Omega_b(\cdot): M_b \rightarrow M$ is a map.

(III.) If $a, b \in L$ such that $a \leq b$, then $\Omega_b m(a) = \frac{m(a)}{m(b)}$.

Then the number $\Omega_b m(a)$ is called the conditional probability of the event a by the condition b in the state m .

Definition. ([10]) Let (L, M) be a q.f.s. and M be a σ -convex set of states. Let $p(\cdot | \cdot): L \times L_c \rightarrow [0, \infty)$ satisfy:

(A) For any $b \in L_c$ and $p(b|b) = 1$.

(B) If $p(c|1) = 1$ and $b, c \star b \in L_c$ then $p(\cdot | b) = p(c \star b | b)p(\cdot | c \star b)$, where $c \star b = (c \vee b^\perp) \wedge b$.

(C) If $a, b, c \in L$, $a \vee b, a \vee b \vee c \in L$ then $p(a|a \vee b \vee c) = p(a|a \vee b)p(a \vee b|a \vee b \vee c)$.

The function $p(\cdot | \cdot)$ will be called a function of conditional probability on L .

(L_c is an additive subset of L)

Theorem. ([10]) Let (L, M) be q.f.s. and M be a σ -convex set of states. Let $m(a) = m(b) = 1$ implies $m(a \wedge b) = 1$. Let $c \in L$,

$$L_c = \{b \in L; b \text{ is not orthogonal to } c\}$$

Let $p(\cdot | \cdot): L \times L_c \rightarrow [0, 1]$ satisfy (A) - (C). Let there be $s(p(\cdot | 1)) = c$. Then for any $b \in L$ $s(p(\cdot | b))$ exists and $s(p(\cdot | b)) = c \star b$. Moreover, if $a \leq b$ ($a \in L$, $b \in L_c$) then

$$p(a|b) = \frac{p(a|1)}{p(b|1)}$$

Conversely, let (L, M) be a supported system and M be a σ -convex set of states. If the function $p^m(\cdot | \cdot)$, $m \in M$ satisfies the axiom I.-III. on $L \times L(m)$ then it satisfies (A) - (C).

These definitions of conditional probabilities do not give an answer for noncompatible elements. In the following we give another definition of conditional states (probabilities) which is equivalent to the previous one on a Boolean algebra.

Let (L, M) be a q.s.f. Let us denote by the symbol $M^* = \{1 - m; \text{ for any } m \in M\}$. If $\Theta \in M^*$ then

(i) $\Theta(0) = 1$ and $\Theta(1) = 0$.

(ii) If $a \perp b$ then $\Theta(a \vee b) = \Theta(a) + \Theta(b) - 1$.

(iii) If $a_1, \dots, a_n \in L$, and moreover they are mutually orthogonal, then

$$\Theta(a_1 \vee \dots \vee a_n) = (\Theta(a_1) + \dots + \Theta(a_n)) - n + 1$$

Definition. Let (L, M) be quantum logic and $M_0 \subset M \cup M^*$. Then M_0 will be called a conditional system of functions if there exists binary operation \ominus on M_0 such that

- (i) For any $\alpha \in M_0$ $\alpha \ominus \alpha \in M_0$.
- (ii) If $\alpha, \beta, \alpha \ominus \beta \in M_0$ then $\alpha \ominus (\alpha \ominus \beta) \in M_0$ and moreover $\alpha \ominus (\alpha \ominus \beta) = \beta$.
- (iii) If $\alpha, \beta, \gamma, \beta \alpha \ominus \beta, \beta \ominus \gamma \in M_0$ then $\alpha \ominus \gamma, (\alpha \ominus \gamma) \ominus (\alpha \ominus \beta) \in M_0$ and moreover

$$(\alpha \ominus \gamma) \ominus (\alpha \ominus \beta) = \beta \ominus \gamma$$

In other words the couple (M_0, \ominus) is a difference set (DS) [11], [12]). It can be shown that a classical Kolmogorovian conditional probability space is DS. Conditional probabilities on quantum logic can also be organized as DS ([17]).

Let $m \in M$ be given such that $m(a) = 1$ iff $a = 1$. It can be shown, that such a state exists and more over there exists a subset M_0 of M such that the set $M^* = M_0 \cup \{\Theta\}$ ($\Theta = 1 - m$) with a partial binary operation \ominus on M^* : $\alpha, \beta \in M^*$, then $\alpha \ominus \beta$ exists iff

- (1) if $\alpha \neq \beta$ then there exists an element $a \in L$ and $\gamma \in M_0$ such that $\beta(a) = 1$, $\gamma(a) = 0$ and $\alpha = \alpha(a)\beta + \alpha(a^\perp)\gamma$ ($\gamma := \alpha \ominus \beta$)
- (2) if $\alpha = \beta$ then $\alpha \ominus \beta = \Theta$.

is DS. It can be shown that:

- (1) if $\alpha \ominus \beta$ exists then it is unique;
- (2) for any $\alpha \in M^*$ $\alpha \ominus \Theta = \alpha$;
- (3) the element $\Theta \ominus \alpha$ exists iff $\alpha = \Theta$

Denote $\beta = p_\alpha(\cdot|a)$. The function p_α has properties of conditional probabilities:

- i for any $a \in L$ such that $p_\alpha(a|1) \neq 0$ $p_\alpha(a|a) = a$
- ii if $c \leq a$ then

$$p_\alpha(c|a) = \frac{\alpha(a \wedge c)}{\alpha(a)}$$

Example 1. Let $L = \{a, a^\perp, b, b^\perp, 0, 1\}$, where $a \perp a^\perp$, $b \perp b^\perp$, $a \vee b = a \vee b^\perp = a^\perp \vee b = a^\perp \vee b^\perp = 1$, $a \wedge b = a \wedge b^\perp = a^\perp \wedge b = a^\perp \wedge b^\perp = 0$ (L is quantum logic) and $m \in M$ such that $m(a) = 0.1$, $m(b) = 0.3$. Let $\alpha, \alpha^*, \beta, \beta^* \in M$ such that $\alpha(a) = 1$, $\alpha^*(a) = 0$, $\beta(b) = 1$, $\beta^*(b) = 0$ and moreover $m = m(a)\alpha + m(a^\perp)\alpha^* = m(b)\beta + m(b^\perp)\beta^*$. From this we get $\alpha(b) = p_m(b|a) \in (0, 1)$, $\alpha^*(b) = p_m(b|a^\perp) \in (\frac{2}{9}, \frac{1}{3})$, $\beta(a) = p_m(a|b) \in (0, \frac{1}{3})$, $\beta^*(a) = p_m(a|b^\perp) \in (0, \frac{1}{7})$. It is clear that $M^* = \{m, \alpha, \alpha^*, \beta, \beta^*, \Theta = 1 - m\}$.

Let $p_m(\cdot|.)$ is the conditional function on L in the state m . Let $a, b \in L$ be such that $p_m(\cdot|a), p_m(\cdot|b)$ exist. If $p_m(a|b) = m(a)$ then we will say that a is independent with

respect to b . In the classical probability space a is independent with respect to b iff b is independent with respect to a . It is not true on quantum logic.

Example 2. Let $L = \{a, a^\perp, b, b^\perp, 0, 1\}$ (L is quantum logic) and $m \in M$ such that $m(a) = 0.1$, $m(b) = 0.3$. Let $\alpha, \alpha^*, \beta, \beta^* \in M$ such that $\alpha(a) = 1$, $\alpha^*(a) = 0$, $\beta(b) = 1$, $\beta^*(b) = 0$ and moreover $m = m(a)\alpha + m(a^\perp)\alpha^* = m(b)\beta + m(b^\perp)\beta^*$. If b is independent to a then $\alpha(b) = \alpha^*(b) = m(b) = 0.3$. It does not imply that a is independent to b . If $\beta(a) = 0.2$, then $\beta^*(a) = \frac{4}{70}$ and $\beta = p_m(\cdot|b)$, $\beta^* = p(\cdot|b^\perp)$.

Note: In the real live it is clear that for example the wheather forecast is dependent on the real weather, but real weather is independent on the forecast. We can say that a conditional state is a measure of dependency of some events.

Futher, systems of states on a quantum logic (and the corresponding algebraic structures) were studied also in the framework of fuzzy sets theory, see e.g. [18],[19],[20],[21].

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DEPT. MATH., SLOVAK UNIVERSITY OF TECHNOLOGY
RADLINSKÉHO 11
813 68 BRATISLAVA
SLOVAKIA
E-mail address: nanasio@cvt.stuba.sk