FUZZY MEASURES AND INTEGRATION

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ABSTRACT. Integration with respect to fuzzy measures is converted into measuring of
special subsets of real plane. By proposed methods the special integrals are covered,
e.g., the Choquet, Sugeno, Choquet-like, PAN and other integrals.

1. INTRODUCTION

The uncertainty modelling by means of classical measures is limited due to the unavoidable additivity of a classical measure. However, in several situations the additivity is either distorted or even completely absent.

Several non-standard theories were proposed and developed so far to overcome the mentioned gap. Recall, e.g., the capacities [2] , belief functions [17, 22, 14], plausibility measures [17, 22, 14], possibility measures [26, 4], submeasures [3, 14], pseudo-additive measures [20, 13, 23, 8, 10], k-order additive measures [5], etc. All these concepts are covered by fuzzy measures introduced by Sugeno [19], see also pre-measures of Šipoš [21].

An essential part of the global measure theory is the corresponding integration theory. However, the standard Lebesgue integral is strongly related to the additivity of the underlying measure. Therefore, any dropping of additivity by measure generalization has excluded the use of the Lebesgue integral, and, consequently, has led to efforts to develop some new types of integrals. Recall, e.g., the Choquet integral [2], the Sugeno integral [19] and its generalizations [16, 24], PAN-integral [25, 22, 1], Choquet-like integrals [12], which all can be applied to arbitrary given fuzzy measure. Special types of (non-additive) integrals are, e.g., the pseudo-additive integrals [20, 13, 8, 10] or g-integrals [15, 11].

An interesting approach to the integration (restricted to the unit interval $[0, 1]$) by means of copulas producing a joint distribution on $[0, 1]^2$ from marginal uniform distributions on $[0, 1]$ was recently proposed by Imaoka [7]. In this paper, we generalize his method in two directions : firstly, we will not restrict the values of measures and functions we will work with, i.e., we will work on $[0, \infty]$ interval in general, and secondly, we will work with more general measures on plane subsets

than Imaoka. Consequently, our approach covers all known fuzzy integrals (and allows to introduce several new types of fuzzy integrals) for which the integral of a basic function $b(a, A) : X \to [0, \infty]$, $a \in [0, \infty]$, $A \in \mathcal{S}$,

$$b(a, A) = \begin{cases} a & \text{if } x \in A \\ 0 & \text{else,} \end{cases}$$

depends only on values a and $m(A)$. Note that among mentioned integrals, the PAN-integral of Yang [25, 22] does not fit the above property , while the PAN-integral of Benvenuti and Vivona [1] fits.

2. FUZZY MEASURES AND INTEGRATION

Let $(X, \mathcal{S})$ be a measurable space. A mapping $m : \mathcal{S} \to [0, \infty]$ is called a fuzzy measure if $m(\emptyset) = 0$ and m is monotone, i.e., $m(A) \leq m(B)$ whenever $A, B \in \mathcal{S}, A \subseteq B$.

Let $f : X \to [0, \infty]$ be a measurable function. It is evident that the system

$$\mathcal{F}_f = (\{x \in X; \ f(x) > t\})_{t \in [0, \infty]}$$

is a measurable decreasing chain (right-continuous, $\mathcal{F}_f(\infty) = \emptyset$) and that $\mathcal{F}_f$ contains a complete information about f. Indeed, for any $x \in X$,

$$f(x) = \sup(t \in [0, \infty]; \ x \in \mathcal{F}_f(t)),$$

with the usual convention $\sup(\emptyset) = 0$. Integration of a function f with respect to a fuzzy measure m is a global evaluation of f. For any given integration method, it depends on f and m only. Equivalently, it depends on $\mathcal{F}_f$ and m only. However, the above argumentation allows us to restrict our information (necessary for integration of f with respect to m) on values of m on the members of the chain $\mathcal{F}_f$, i.e., on the function $h(f, m) : [0, \infty] \to [0, \infty]$ given by

$$h(f, m)(t) = m(\mathcal{F}_f(t)) = m(\{x \in X; \ f(x) > t\}).$$

The monotonicity of m and $\mathcal{F}_f$ yields the monotonicity of $h(f, m)$, i.e., $h(f, m)$ is non-increasing. More, $h(f, m)$ determines its hypergraph

$$H(f, m) = \{(u, v) \in [0, \infty]^2; \ v < h(f, m)(u)\}.$$

Note that $H(f, m)$ is a Borel subset of $[0, \infty]^2$, see also [7] when restricted to $[0, 1]$ values. All mentioned facts lead us to the following conclusion : an integral of a non-negative measurable function f with respect to a fuzzy measure m is some measure of the corresponding hypergraph $H(f, m)$.

Definition 1. Let M be a given fuzzy measure on $\mathcal{B}([0,\infty]^2)$. For an arbitrary given measurable space $(X,\mathcal{S})$, a measurable function $f : X \to [0,\infty]$ and a fuzzy measure $m : \mathcal{S} \to [0,\infty]$, a general fuzzy integral $(M) - \int_X f\, dm$ given by

$$(M) - \int_X f\, dm = M\left(H(f,m)\right)$$

will be called an M-fuzzy integral.

3. EXAMPLES

It is evident that the properties of an M-fuzzy integral are related to the properties of the fuzzy measure M. A deeper investigation of this topic is a matter of further research. Now, we give only some examples of M-fuzzy integrals.

Example 1. Let $C = \lambda_2$ be the Lebesgue measure on Borel subsets of $[0,\infty]^2$. Then C-fuzzy integral is just the Choquet integral [2]. λ_2 is a σ-additive measure on $\mathcal{B}([0,\infty]^2)$ and on $\mathcal{B}([0,1]^2)$ it is covered by Imaoka [7].

Example 2. Let $S(B) = \lambda(\{x \in [0,\infty]; (x,x) \in B\}), B \in \mathcal{B}([0,\infty]^2), \lambda$ the Lebesgue measure on $\mathcal{B}([0,\infty])$. Then the corresponding S-fuzzy integral is just the Sugeno integral [19], more precisely, the generalization of Ralescu and Adams [16]. S is a σ-additive measure and again when restricted to $\mathcal{B}([0,1]^2)$, it is covered by Imaoka [7].

Example 3. For $B \in \mathcal{B}([0,\infty]^2)$, let

$$Sh(B) = \sup(uv;\ [0,u] \times [0,v] \subset B).$$

Then the Sh-fuzzy integral is just the Shilkret integral [18]. Sh is not a σ-additive measure and it cannot be reached (when restricting to $\mathcal{B}([0,1]^2)$) by Imaoka's approach.

Example 4. For $B \in \mathcal{B}([0,\infty]^2)$, let

$$M(B) = \text{diam}(B) = \sup(d(P,Q);\ P,Q \in B).$$

The corresponding M-fuzzy integral is a new type of fuzzy integral. Note that

$$(M) - \int_X b(a,A)\, dm = \sqrt{a^2 + m(A)^2},$$

whenever $a > 0$, $m(A) > 0$. Otherwise, $b(a, A)$ is identicaly equal to zero, the corresponding surface $H(b(a, A), m)$ is empty set and hence

$$(M) - \int_X b(a, A)\ dm = 0.$$

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