SENSITIVITY OF FUZZY CONNECTIVES

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Abstract. The notion of modulus of continuity was introduced in real analysis by A. Lebesgue in 1910 although this notion was known earlier. Using this concept in connection with membership functions or fuzzy logical connectives the notion of sensitivity was introduced in [1]. In this paper some remarks and estimation on measure of sensitivity of fuzzy logical connectives are done and some open questions arise.

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Introduction

The modeling of fuzzy concepts through the assignment of membership function as well as choice of a fuzzy logic are subjective and depends on various factors [2],[3]. In specific situations, a sensitivity of used connectives can be an important factor. We shall use the notion of a sensitivity that is motivated by the notion of modulus of continuity of real functions introduced by A. Lebesgue in 1910.

Definition 1. For any mapping $f : \mathrm{R}^n \to \mathrm{R}$, and for $\delta \geq 0$, and $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$, let

$$\rho_f(\delta) = \bigvee_{|x_i - y_i| \leq \delta} |f(x) - f(y)|$$

The function $\rho_f : [0,\infty] \to [0,\infty]$ is an *extreme measure of sensitivity* of f (shortly *sensitivity of f*). We say that f is less sensitive than g if for all $\delta \geq 0$ $\rho_f(\delta) \leq \rho_g(\delta)$.

Preliminaries

Definition 2. A unary operator $n : [0;1] \to [0;1]$ is called a *negator* if for any a,b in [0,1]

a) $a \leq b \Rightarrow n(a) \geq n(b)$

b) $n(0) = 1, n(1) = 0$

A negator n is called a *strict* if it is a permutation. A strict negator n is called *strong* if it is *involutive,* i.e. for any x $\in [0;1]$ n(n(x)) = x. It can be easily proved that for any strict negator n, its inverse n^{-1} is also a strict negator, and both are continuous.

Here are some examples of negators on [0;1].

Example 2.

1) $n_1(x) = 1 - x$ *is an strong negator*

2) $n_2(x) = 1 - x^2$ *is a strict non involutive negator*

Definition 3. A binary operation $T : [0;1]^2 \to [0;1]$ is called a *t-norm* if for any $a,b,c \in [0;1]$

$T(1,a) = a$ *(boundary condition)*

$a \le b \Rightarrow T(a,c) \le T(b,c)$ *(monotonicity)*

$T(a,b) = T(b,a)$ *(commutativity)*

$T(T(a,b),c)) = T(a,T(b,c))$ *(associativity)*

A continuous t-norm T is called *Archimedean* if $T(x,x) < x$ for any $x \in (0;1)$. A t-norm T is called *strict* if $T(a,c)<T(b,c)$ for any $0 \le a < b \le 1,\ 0 < c \le 1$. A continuous Archimedean non strict t-norm is called a *nilpotent* t-norm.

Here are some examples of t-norms on $[0;1]^2$.

Example 3.

1) $T_M(x,y) = \min(x,y)$

2) $T_P(x,y) = xy$

3) $T_L(x,y) = \max(0,x+y-1)$

4) $T_W(x,y) = \begin{cases} \min(x,y) & \text{if } \max(x,y) = 1 \\ 0 & \text{otherwise} \end{cases}$

Let $\{(a_k,b_k),\ k \in K\}$ be a family of pairwise disjoint subintervals of $[0;1]$ and $\{T_k,\ k \in K\}$ be a family of t-norm different from T_M. Then the *ordinal sum* $\{[a_k,b_k],T_k\}_{k\in K}$

$$T(x,y) = \begin{cases} a_k + (b_k - a_k)T_k\left(\dfrac{x-a_k}{b_k-a_k}, \dfrac{y-a_k}{b_k-a_k}\right) & \text{if } x,y \in [a_k,b_k] \\ \min(x,y) & \text{otherwise} \end{cases}$$

Definition 4. A binary operation $S : [0;1]^2 \to [0;1]$ is called a *conorm* if for any $a,b,c \in [0;1]$

$S(0,a) = a$ *(boundary condition)*

$a \le b \Rightarrow S(a,c) \le S(b,c)$ *(monotonicity)*

$S(a,b) = S(b,a)$ *(commutativity)*

$S(S(a,b),c)) = S(a,S(b,c))$ *(associativity)*

If n is an involutive negator and T is a t-norm then the binary operation

$$S(a,b) = n(T(n(a),n(b)))$$

is a t-conorm. Then T,S,n fulfill the generalized De Morgan laws $n(S(a,b)) = T(n(a),n(b))$ or, equivalently, $n(T(a,b)) = S(n(a),n(b))$

In this case we say that (T,S,n) is a *De Morgan triple* or dual triple.

Results

Let us show the extreme measure of sensitivity for some fuzzy connectives (negators, t-norms and t-conorms). Similar problems can be solved for other connectives e.g. implicators, which are considered in [4]. For connectives the sensitivity is a mapping $\rho_f:[0,1]\to[0,1]$.

Example 4.
$\rho_{n_1}(\delta)=\delta$ and $\rho_{n_2}(\delta)=2\delta-\delta^2$ for $\delta\in[0,1]$, where n_1 and n_2 are negators in Example 2.

$\rho_{T_M}(\delta)=\delta$, $\rho_{T_P}(\delta)=2\delta-\delta^2$, $\rho_{T_L}(\delta)=min(1,2\delta)$, $\rho_{T_W}(\delta)=\begin{cases}0 & for\quad \delta=0\\ 1 & otherwise\end{cases}$ for $\delta\in[0,1]$, where $\rho_{T_M},\rho_{T_P},\rho_{T_L}$ and ρ_{T_W} are t-norms in Example 3.

Example 5.
Let n be an involutive negator then Fodor nilpotent minimum is defined as follows

$$T(x,y)=\begin{cases}min(x,y) & if\quad y>n(x)\\ 0 & otherwise\end{cases}$$

then

$\rho_T(\delta)=Min(1,x_0+\delta)$, where x_0 is the equilibrium of n, i.e. a solution of $n(x)=x$ (see Figure 1).

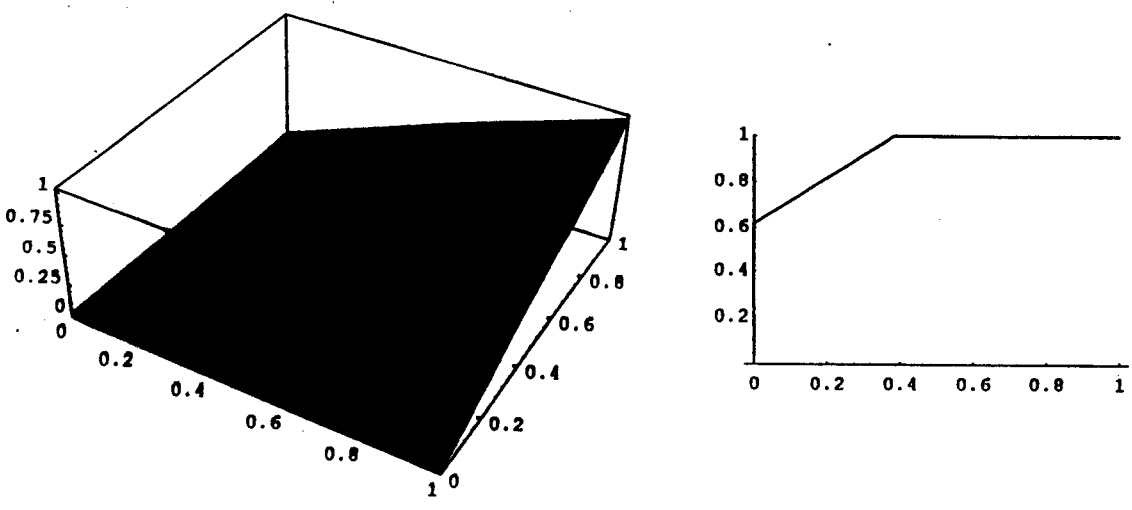

Figure 1. Fodor nilpotent minimum for $n(x)=\sqrt{1-x^2}$ and its sensitivity

Example 6.

Let $T = \{[1/2,3/4], T_W\}$ is ordinal sum then $\rho_T(\delta) = \begin{cases} 0 & \delta = 0 \\ 1/4+\delta & 0<\delta<1/4 \\ 1/2 & 1/4<\delta<\langle 1/2 \\ \delta & 1/2 \le \delta \le 1 \end{cases}$

For the following two theorems see also [1].

Theorem 1. If t-norm T and t-conorm S are dual with respect to n (x) = 1-x then they have the same sensitivity.

Theorem 2. The t-norm $T_M(x,y) = \min(x,y)$, t-conorm $S_M(x,y) = \max(x,y)$, and negator $n_I(x)$ are the least sensitive among all t-norms, t-conorms, and negators, respectively.

Corollary 1. $\rho_F(\delta) \ge \delta$, $\delta \in [0;1]$ where F is t-norm, t-conorm or negator respectively.

Let (T,S,1-x) be a *De Morgan triple.* For all t-norms in Example 3 we have

$$\rho_T(\delta) = \rho_S(\delta) = S(\delta,\delta).$$

Let us consider t-norm T in Example 6, then we have $\rho_T(\delta) = \rho_S(\delta) \ne S(\delta,\delta)$. In connection with this the following problem arises.

Problem 1. To characterize the De Morgan triple (T,S,1-x) for which the following equality is valid $\rho_T(\delta) = \rho_S(\delta) = S(\delta,\delta)$.

A t-norm T is a *copula* iff $T(x_1,x_2) + T(y_1,y_2) \ge T(x_1,y_2) + T(y_1,x_2)$ for all $x_1,x_2,y_1,y_2 \in [0;1]$ with $x_1 \langle y_1$, $x_2 \langle y_2$.

The following theorem gives an estimation for sensitivity of a t-norm T that is a copula [5].

Theorem 3. Let T be a t-norm and copula then we have $\delta \le \rho_T(\delta) \le 2\delta$.

Proof: The first part of the inequality is Corollary 1. Since T is a copula ,these yield

$$|T(a,y) - T(a,x)| \le |T(1,y) - T(1,x)|, \quad |T(y,b) - T(x,b)| \le |T(y,1) - T(x,1)|$$

for all $a,b,x,y \in [0;1]$.

Applying this twice we obtain

$$\rho_T(\delta) = \bigvee_{|x_i-y_i| \le \delta} |T(x_1,x_2) - T(y_1,y_2)| = \bigvee_{|x_i-y_i| \le \delta} |T(x_1,x_2) - T(x_1,y_2) + T(x_1,y_2) - T(y_1,y_2)| \le$$

$$\le \bigvee_{|x_i-y_i| \le \delta} \left(|T(x_1,x_2) - T(x_1,y_2)| + |T(x_1,y_2) - T(y_1,y_2)| \right) \le$$

$$\le \bigvee_{|x_i-y_i| \le \delta} \left(|T(1,x_2) - T(1,y_2)| + |T(x_1,1) - T(y_1,1)| \right) \le \bigvee_{|x_i-y_i| \le \delta} \left(|x_2 - y_2| + |x_1 - y_1| \right) \le 2\delta$$

It is obvious that the sensitivity of the ordinal sum depends not only on the family of used t-norms but also on their location (see the following example).

Example 7. Consider the following ordinal sums (see Figure 2)
$T_1(x,y)=\{[0,1/2],T_P\}$, $T_2(x,y)=\{[1/4,1/2],T_P\}$, $T_3(x,y)=\{[1/2,1],T_P\}$, where $T_P(x,y)=xy$

then

$$\rho_{T_1}(\delta)=\begin{cases} 2\delta-2\delta^2 & \delta\le 1/4 \\ 1/8+\delta & 1/4<\delta\le 3/4 \\ -1+4\delta-2\delta^2 & \textit{otherwise} \end{cases}$$

$$\rho_{T_2}(\delta)=\begin{cases} 2\delta-2\delta^2 & \delta\le 1/4 \\ \delta+1/8 & 1/4<\delta\le 1/2 \\ 3/4-2(\delta-3/4)^2 & 1/2<\delta<3/4 \\ \delta & \text{otherwise} \end{cases}$$

$$\rho_{T_3}(\delta)=\begin{cases} 2\delta-2\delta^2 & \delta\le 1/2 \\ \delta & \text{otherwise} \end{cases}$$

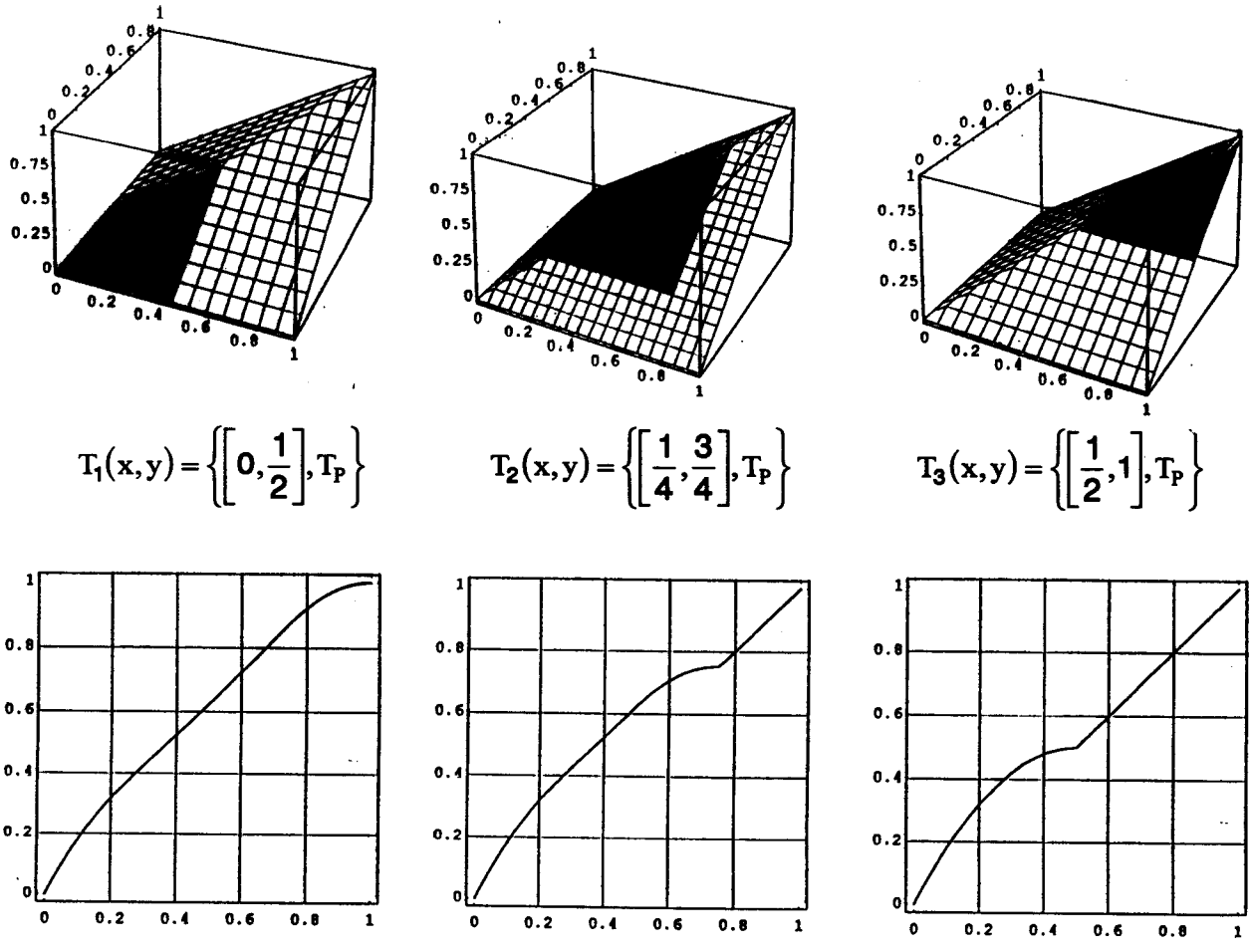

Figure 2. Ordinal summs with the same component and their sensitivities

It is known that any Archimedean continuous t-norm is an ordinal sum of continmuos t-norms. Thus, it is conveniet to ask for the following problem.

Problem 2. Find the relation between sensitivity of the ordinal sum and sensitivities of their components.

The following theorem gives a lower estimation of sensitivity of the ordinal sum respect to sensitivities of their components.

Theorem 4. Let T(x,y) = $\{[a_k, b_k], T_k\}_{k\in K}$ be an ordinal sum of t-norms. Then

$$\rho_T(\delta) \geq \sup_{k\in K}\left((b_k - a_k)\rho_{T_k}\left(\min\left(1, \frac{\delta}{b_k - a_k}\right)\right)\right)$$

Proof. If $\delta \geq b_k - a_k$ for some k $\in$ K then using Corollary 1 we have

$$(b_k - a_k)\rho_{T_k}\left(min\left(1, \frac{\delta}{b_k - a_k}\right)\right) = b_k - a_k \leq \delta \leq \rho_T(\delta)$$

Let now $\delta < b_i - a_i$ for some i $\in$ K

Take $(x_1, y_1), (x_2, y_2) \in [a_i, b_i]^2$ *such that* $0 \leq x_2 - x_1 \leq \delta, 0 \leq y_2 - y_1 \leq \delta$

Then $0 \leq \frac{y_2 - y_1}{b_i - a_i} \leq \frac{\delta}{b_i - a_i}$, $0 \leq \frac{x_2 - x_1}{b_i - a_i} \leq \frac{\delta}{b_i - a_i}$. Moreover we can choose such $(x_1, y_1), (x_2, y_2)$ that for arbitrary $0 < \varepsilon$ we have

$T_i\left(\frac{x_2 - a_i}{b_i - a_i}, \frac{y_2 - a_i}{b_i - a_i}\right) - T_i\left(\frac{x_1 - a_i}{b_i - a_i}, \frac{y_1 - a_i}{b_i - a_i}\right) \geq \rho_{T_i}\left(\frac{\delta}{b_i - a_i}\right) - \varepsilon$. It implies

$$T(x_2, y_2) - T(x_1, y_1) = (b_i - a_i)\left(T_i\left(\frac{x_2 - a_i}{b_i - a_i}, \frac{y_2 - a_i}{b_i - a_i}\right) - T_i\left(\frac{x_1 - a_i}{b_i - a_i}, \frac{y_1 - a_i}{b_i - a_i}\right)\right) \geq$$

$$\geq (b_i - a_i)\left(\rho_{T_i}\left(\frac{\delta}{b_i - a_i}\right) - \varepsilon\right)$$

Therefore $\rho_T(\delta) \geq T(x_2, y_2) - T(x_1, y_1) \geq (b_i - a_i)\left(\rho_{T_i}\left(\frac{\delta}{b_i - a_i}\right) - \varepsilon\right)$

or

$$\rho_T(\delta) \geq (b_i - a_i)\left(\rho_{T_k}\left(\frac{\delta}{b_i - a_i}\right)\right)$$

Remark. For T_2 in Example 7 the equality

$$\rho_T(\delta) = \delta \vee \sup_{k\in K}\left((b_k - a_k)\rho_{T_k}\left(\min\left(1, \frac{\delta}{b_k - a_k}\right)\right)\right), \quad \delta \in [0,1]$$

holds. It evokes the next problem.

Problem 3. To characterize such ordinals sums for which the equality

$$\rho_T(\delta) = \delta \vee \sup_{k \in K}\left((b_k - a_k)\rho_{T_k}\left(\min\left(1, \frac{\delta}{b_k - a_k}\right)\right)\right), \quad \delta \in [0,1]$$

holds.

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