

FUZZY PREFERENCE STRUCTURES AND t-REVERSIBLE t-NORMS

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Abstract. Some possibilities of constructions of fuzzy preference structures using continuous t-norms are studied and the role of t-reversible t-norms is discussed.

Key words: fuzzy preference structure, t-norm

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Introduction

A preference structure is a basic concept of a preference modeling. In a classical preference structure, a decision-maker gives three decisions for any pair (a,b) from a set of alternatives A . His decisions define a triplet (P,I,J) of crisp binary relations on A :

- 1) a preferred to $b \quad \Leftrightarrow (a,b) \in P$ (*strict preference*)
- 2) a and b are indifferent $\Leftrightarrow (a,b) \in I$ (*indifference*)
- 3) a and b are incomparable $\Leftrightarrow (a,b) \in J$ (*incomparability*)

For any binary operations B on A , we denote by B^{-1} its *inverse* i.e. $(a,b) \in B^{-1} \Leftrightarrow (b,a) \in B$, B^c its *complement* i.e. $(a,b) \in B^c \Leftrightarrow (a,b) \notin B$, B^d its *dual* i.e. $B^d = (B^{-1})^c = (B^c)^{-1}$. We denote: $B(a,b) = 1 \Leftrightarrow (a,b) \in B$ and $B(a,b) = 0 \Leftrightarrow (a,b) \notin B$

Definition 1. A preference structure (PS) on a set A is a triplet (P,I,J) of binary relations on A such that

- (PS1) I, J are symmetric i.e. $I(a,b) = I(b,a)$, $J(a,b) = J(b,a)$
- (PS2) I is reflexive, J is irreflexive i.e. $I(a,a) = 1$, $J(a,a) = 0$
- (PS3) $P \cap I = P \cap J = P \cap P^{-1} = I \cap J = \emptyset$
- (PS4) $P \cup P^{-1} \cup I \cup J = A \times A$

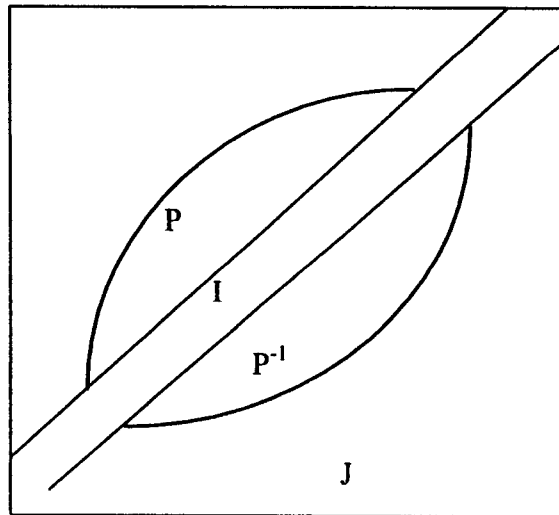


Figure 1. Partition of $A \times A$

Define $R = P \cup I$ called the *large preference relation* (or characteristic relation). It can be easily proved that $R^d = P \cup J$ and

$$P = R \cap R^d, \quad I = R \cap R^{-1}, \quad J = R^c \cap R^d$$

It allows to construct a preference structure (P, I, J) from one reflexive binary operation R only.

Example 1. (we are unable to compare the first and the third alternative)

$$\text{If } R = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \text{ then } R^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad R^c = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \quad R^d = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$P = R \cap R^d = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad I = R \cap R^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad J = R^c \cap R^d = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Fuzzy preference structure

Rigidity of classical binary preferences has caused trouble a decision process for a long time. In real situations it is necessary to consider a strength of preference rather than yes-no preference, even if the individual decisions are yes-no. Therefore, a natural demand has led to the introduction of a fuzzy preference structure. It is natural to replace the notions used in the Definition 1 by their fuzzy equivalents. We shall see that fuzzification of (PS4) can cause some troubles. Problems consist in the fact that the axiom (PS4) in the classical case can be expressed by many equivalent ways e.g.

$$A1) (P \cup I)^c = P^{-1} \cup J$$

$$A2) J = P^c \cap P^d \cap I^c$$

$$A3) P \cup P^{-1} \cup I \cup J = A \times A$$

We suppose that the reader familiar with the notions of fuzzy set, t-norm, t-conorm, ordinal sum of t-norms, Frank t-norm, fuzzy relation, fuzzy negation, De Morgan triplet (see e.g. [1,5]).

Consider a continuous De Morgan triplet (T, S, N) and a triplet of fuzzy relations (P, I, J) . Then we fuzzify the axioms from the Definition 1 and (A1) - (A3)

$$(FPS1) \quad I, J \text{ are symmetric i.e. } I(a, b) = I(b, a), \quad J(a, b) = J(b, a)$$

$$(FPS2) \quad I \text{ is reflexive, } J \text{ is irreflexive i.e. } I(a, a) = 1, \quad J(a, a) = 0$$

$$(FPS3) \quad T(P(a, b), I(a, b)) = T(P(a, b), J(a, b)) = T(I(a, b), J(a, b)) = T(P(a, b), P^{-1}(a, b)) = 0$$

$$(FA1) \quad N(S(P, I)) = S(P^{-1}, J)$$

$$(FA2) \quad J = T(P^c, P^d, I^c)$$

$$(FA3) \quad S(P, P^{-1}, I, J) = 1$$

If we use a positive continuous t-norm T (i.e. $T(a, b) = 0 \Rightarrow a = 0$ or $b = 0$) and (P, I, J) is a triplet of fuzzy relations satisfying (FS1), (FS2), (FS3). Then, it can be proved [1,2,4] that under any of the conditions (FA1), (FA2), (FA3) the relations P, I, J are crisp. Moreover, if T is a non Archimedean t-norm with zero divisors then non-trivial values of P, I, J are restricted in some interval $[0, r)$, $r < 1$. This strongly recommends to use a nilpotent t-norm only. Because any nilpotent

t-norm is isomorphic to the Lukasiewicz t-norm T_L , we shall use the Lukasiewicz triplet (T_L, S_L, N) only, where

$$T_L(a, b) = \max(0, a + b - 1), \quad S_L(a, b) = \min(1, a + b), \quad N(a) = 1 - a \quad \text{for any } a, b \in [0, 1]$$

In this case, it was proved [1,4] that from validity (FA1) and $T_L(P(a,b), I(a,b)) = T_L(P(a,b), J(a,b)) = 0$ follows (FA2), (FA3) and $T_L(I(a,b), J(a,b)) = T_L(P(a,b), P^{-1}(a,b)) = 0$. Now we can define Lukasiewicz preference structure.

Definition 2. A triplet (P, I, J) of fuzzy relations on A is called Lukasiewicz preference structure (shortly LPS) on A if and only if the following conditions are satisfied for any $a, b \in [0, 1]$

(LPS1) I, J are symmetric i.e. $I(a, b) = I(b, a)$, $J(a, b) = J(b, a)$

(LPS2) I is reflexive, J is irreflexive i.e. $I(a, a) = 1$, $J(a, a) = 0$

(LPS3) $T_L(P(a, b), I(a, b)) = T_L(P(a, b), J(a, b)) = 0$

(LPS4) $N(S_L(P(a, b), I(a, b))) = S(P^{-1}(a, b), J(a, b))$

Remark. (LPS3), (LPS4) together are equivalent to

$$P(a, b) + P^{-1}(a, b) + I(a, b) + J(a, b) = 1 \quad a, b \in [0, 1]$$

The idea of using some reflexive large preference relations R to construct FPS can be fuzzified too. A natural way is to consider a continuous De Morgan triplet (T, S, N) and to define

$$P(a, b) = T(R(a, b), N(R(a, b)))$$

$$I(a, b) = T(R(a, b), R(b, a))$$

$$J(a, b) = T(N(R(a, b)), N(R(b, a)))$$

such that $R = S(P, I)$ and $R^d = S(P, J)$

It was proved [1,3] that such construction is possible in the crisp case only. Therefore, we define FPS generated by a reflexive binary relation R (FPS- R) as follows. Let (T, S, N) be a continuous De Morgan triple and R be a reflexive fuzzy binary operation on A . To define fuzzy binary operations P, I, J we propose the following axioms:

(R) there exist three functions $p, i, j : [0, 1]^2 \rightarrow [0, 1]$ such that

$$P(a, b) = p(R(a, b), N(R(a, b)))$$

$$I(a, b) = i(R(a, b), R(b, a))$$

$$J(a, b) = j(N(R(a, b)), N(R(b, a)))$$

(R2) p, i, j are non decreasing with respect to both arguments

(R3) i, j are symmetric functions

(R4) $R = S(P, I)$ and $R^d = S(P, J)$

Thus, such FPS- R is described by (p, i, j, T, S, N)

Theorem 1 [1,3] Suppose (p, i, j, T, S, N) satisfies (R1)-(R4). Then T is nilpotent t-norm and for all $x, y \in [0, 1]$

$$T(x, y) \leq p(x, y), \quad i(x, y), j(x, y) \leq \min(x, y)$$

Again, nilpotent t-norms are recommended. Because any nilpotent t-norm is isomorphic to the Lukasiewicz t-norm T_L , we shall use the Lukasiewicz triplet (T_L, S_L, N) only. In this case we can (R4) read in the form:

$$p(x, 1-y) + i(x, y) = x, \quad p(x, 1-y) + j(1-x, 1-y) = 1-y \quad x, y \in [0, 1] \quad (1)$$

Theorem 2. [1,3] Consider (T_L, S_L, N) and a reflexive fuzzy binary relation R on A . Assume that (p, i, j) satisfies (R1)-(R4) and (P, I, J) is defined via (R1). Then (P, I, J) is LPS

Theorem 3. [1,3] Consider (T_L, S_L, N) and a reflexive fuzzy binary relation on A and a reflexive fuzzy binary R on A . Assume that p, i, j are continuous t-norms Then (P, I, J) defined via (R1) satisfies (R1)-(R4) if and only if for $x, y \in [0, 1]$ $p(x, y) = T^{1/s}(x, y)$, $i(x, y) = T^s(x, y) = j(x, y)$, where T^s and $T^{1/s}$ are Frank t-norms [1,5]

In other words, if we want to have p, i, j continuous t-norms then we must use Frank t-norms. If we want to use one t-norm only, then we must use $T^1(x, y) = xy$.

FPS and t-reversible t-norm

Given t-norm T , the binary operation

$$T^*(x, y) = \max(0, x+y-1+T(1-x, 1-y))$$

is called t-reverse of t-norm T [5]. Moreover $T \geq T_L$ if and only if

$$T^*(x, y) = x+y-1+T(1-x, 1-y)$$

A t-norm is called t-reversible if T^* is t-norm too. It is known that a continuous t-norm T , $T \geq T_L$ is reversible if and only if T is a Frank t-norm or ordinal sum of Frank t-norms.

Theorem 4. Consider Lukasiewicz De Morgan triplet (T_L, S_L, N) and FPS-R defined via (R1)-(R4). Assume that i, j are continuous t-norms. Then $i \geq T_L$, $j \geq T_L$ and $i^* = j$ and $j^* = i$.

Proof. The equations (1) imply $i(x, y) = x+y-1+j(1-x, 1-y)$ and $j(x, y) = x+y-1+i(1-x, 1-y)$

Theorem 5. Consider Lukasiewicz De Morgan triplet (T_L, S_L, N) . Assume that i, j are continuous t-norms. Then (P, I, J) defined via (R1)-(R4) is FPS-R if and only if i is a Frank norm or an ordinal sum of Frank norms, $j = i^*$, and p is defined by $p(x, y) = x - i(x, 1-y)$, $x, y \in [0, 1]$

Proof. The implication from left to right follows from Theorem 4. To prove the opposite direction it is sufficient to show that p is non decreasing. The equation $p(x, y) = x - i(x, 1-y)$ implies that p is non decreasing in the second argument and the equation $p(x, y) = y - j(1-x, y)$ $x, y \in [0, 1]$ implies that p is non decreasing in the first argument.

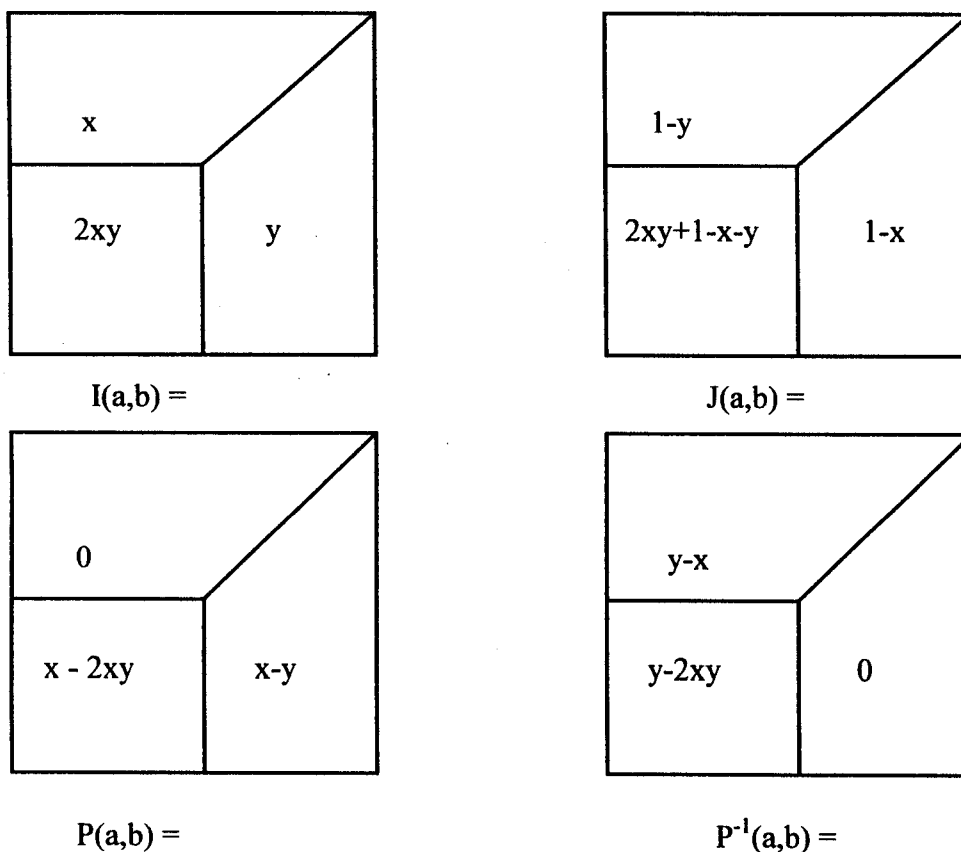
Example 3. Put

$$i(x, y) = \begin{cases} 2xy & x, y \in [0, \frac{1}{2}] \\ \min(x, y) & \text{otherw.} \end{cases}$$

then

$$j(x, y) = i^*(x, y) = \begin{cases} \frac{1}{2} + 2(x - \frac{1}{2})(y - \frac{1}{2}) & x, y \in [0, \frac{1}{2}] \\ \min(x, y) & \text{otherw.} \end{cases} \quad p(x, y) = x - i(x, 1-y)$$

The R-FPS is described on Picture2 with $x = R(a, b)$, $y = R(b, a)$, $P, I, J : [0, 1]^2 \rightarrow [0, 1]$



Picture 2.

References

- [1] J.Fodor, *Preference modeling using fuzzy sets*, Plenaries and tutorials, EURO/INFORMS-97 Barcelona .
- [2] C.Alsina, *On a family of connectives for fuzzy sets*, Fuzzy sets and systems, 16(1985) 231-235
- [3] J.Fodor and M.Roubens, *Fuzzy preference modeling and Multicriteria Decision Support*, Kluwer, Dordrecht, 1994
- [4] B.Van de Walle, B.De Baets and E.Kerre, *A comparative study of completeness conditions in fuzzy preference structures*, Proc. IFSA'97, Prague, vol III, 74-79.
- [5] M.Šabo, *On t-reverse of t-norms*, Tatra Mt.Math.Publ.12(1997),35- 40

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