ADDITIVE GENERATORS AND DISCONTINUITY

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Abstract. We discuss some necessary conditions of associativity of generated two-place functions, when corresponding additive generators are discontinuous. There is described necessary and sufficient condition of associativity of generated function, when corresponding generator has only one point of discontinuity. At the end of this paper some examples are illustrated.

1. Introduction

First we recall the definitions of some concepts (see [4], [6]).

Definition 1. A triangular norm (briefly t-norm) is a function $T\colon [0,1]^2 \to [0,1]$ such that

$$\begin{aligned} &T(x,y) = T(y,x) && \text{(commutativity)}\\ &T(T(x,y),z) = T(x,T(y,z)) && \text{(associativity)}\\ &x \le y \Rightarrow T(x,z) \le T(y,z) && \text{(monotonicity)}\\ &T(x,1) = x && \text{(boundary conditon).} \end{aligned}$$

Definition 2. Let $f\colon [0,1] \to [0,\infty]$ be a non-increasing function. Then the function $f^{(-1)}\colon [0,\infty] \to [0,1]$ defined by

$$f^{(-1)}(y) = \sup\{x \in [0,1] \mid f(x) > y\}$$

is called the *pseudo-inverse of function f*.

Definition 3. Let $f\colon [0,1] \to [0,\infty]$ be a strictly decreasing function, $f(1) = 0$ and let the function $T\colon [0,1]^2 \to [0,1]$ be given by formula

$$T(x,y) = f^{(-1)}(f(x) + f(y)) \ \forall x,y \in [0,1] \tag{1}$$

where $f^{(-1)}$ is the pseudo-inverse of the function f. Function f is called a *conjunctive additive generator of the function T*.

Definition 4. Let $f\colon [0,1] \to [0,\infty]$ be a non-increasing function. The range of the function f is *relatively closed under the addition* if and only if for all $x, y \in [0,1]$ we have $f(x)+f(y) \in Ran(f)$ or $f(x)+f(y) \geq \lim_{x\to 0^+} f(x)$.

Next theorem was introduced by Klement, Mesiar and Pap in [4].

Theorem 1. *Let $f\colon [0,1] \to [0,\infty]$ be a conjunctive additive generator of T with Ran(f) relatively closed under the addition. Then the function T is a t-norm.*

2. The points of discontinuity of additive generator

Let f be a conjunctive additive generator of T. For $a \in [0,1]$, denote $\lim_{x\to a^-} f(x) = f(a_-)$, $\lim_{x\to a^+} f(x) = f(a_+)$ and $D_f = D_f(0,1) = \{a \in (0,1) \mid f \text{ is discontinuous at } a\}$. If $a \in D_f(0,1)$, then $2f(a_+) \leq f(a_+) + f(a) \leq 2f(a_-)$. We devide the set D_f into four disjoint subsets.
$D_0(f) = \{a \in (0,1) \mid 2f(a) \leq f(a_-)\}$,
$D_1(f) = \{a \in (0,1) \mid f(a)+f(a_-) \leq f(a_-) < 2f(a)\}$,
$D_2(f) = \{a \in (0,1) \mid 2f(a_+) \leq f(a_-) < f(a)+f(a_+)\}$,
$D_3(f) = \{a \in (0,1) \mid f(a_-) < 2f(a_+)\}$.
Then $D_f = D_0(f) \cup D_1(f) \cup D_2(f) \cup D_3(f)$, and $D_i(f) \cap D_j(f) = \emptyset$ for all $i, j \in \{0,1,2,3\}$, $i \neq j$.
Let
$L_-(f) = \{u \in R^+ \mid \exists t \in (0,1]\, u = f(t_-)\}$
$H_-(f) = \{u \in R^+ \mid \exists a \in D_f(0,1)\, u = f(a_-) - f(a)\}$.

Full proofs of the next results can be found in [8].

Theorem 2. *Let $f\colon [0,1] \to [0,\infty]$ be a conjunctive additive generator of T. If T is a t-norm, then $H_-(f) \cap L_-(f) = \emptyset$.*

Now, we will investigate the set D_f.

Theorem 3. *Let $f\colon [0,1] \to [0,\infty]$ be a conjunctive additive generator of T.
If T is a t-norm, then $D_1(f) = \emptyset$.*

Theorem 4. *Let $f\colon [0,1] \to [0,\infty]$ be a conjunctive additive generator of T which is left-continuous at point 1.
If T is a t-norm, then $D_2(f) = \emptyset$.*

Previous two theorems imply Corollary 1.

Corollary 1. *Let $f\colon [0,1] \to [0,\infty]$ be a conjunctive aditive generator of T which is left-continuous at point 1.*
If T is a t-norm, then $D_f = D_0(f) \cup D_3(f)$.

Now, we will discuss the case a conjunctive additive generator of T when f has only one point of discontinuity $a \in [0,1]$. If $a = 0$, then T is a t-norm, because the value of function f at point 0 has no influence on generated function T, i.e., f and g, where $g/(0,1] = f$ and $g(0) = f(0_+)$, generate the same function T. If a=1, then f has the range relatively closed under addition and hence T is a t-norm (see [5]). Next theorem covers all situations, whenever $a \in (0,1)$.

Theorem 5. *Let $f\colon [0,1] \to [0,\infty]$ be a conjunctive additive generator of T and let $a \in (0,1)$ be the only point of discontinuity of f.*
Then T is a t-norm if and only if $2f(a) \leq f(a_-)$ and $f(0_+) - f(a_-) \leq f(a_+)$.

Example 1. Let $a \in (0,1)$, $b \geq 1-a$, $v \geq 0$ and let $f_{a,b,v}\colon [0,1] \to [0,\infty]$ be defined by the following formula

$$f_{a,b,v}(x) = \begin{cases} 1-x & \text{if } a < x \leq 1 \\ b & \text{if } x = a \\ a+b+v-x & \text{if } 0 \leq x < a. \end{cases}$$

It is obvious that $f_{a,b,v}(x)$ is a conjunctive additive generator, f is discontinuous only at point a, $a \in (0,1)$, $b = f(a)$, and $v = f(a_-) - f(a)$. By Theorem 5, $f_{a,b,v}$ is an additive generator of a t-norm if and only if $a \leq \frac{1}{2}$ and $v \geq b$.

If $a = \frac{1}{2}$ and $v \geq b$, then the corresponding t-norm T is the same (see [7], Example 1). This function T is defined by

$$T(x,y) = \begin{cases} 0 & \text{if } x+y < 1 \\ \frac{1}{2} & \text{if } (x,y) \in M \\ x+y-1 & \text{otherwise.} \end{cases}$$

where $M = \{(x,y) \in [0,1]^2 \mid \frac{1}{2} \leq x,y \leq 1 \text{ and } x+y \leq \frac{3}{2}\}$.

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