

A NOTE ON NON-CONTINUOUS T-NORMS

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ABSTRACT. Multiplicative generator φ of a triangular norm is a special monotone function $\varphi : [0, 1] \rightarrow [0, 1]$ with fixed point 1 and $\varphi(0) < 1$. The corresponding t-norm T is defined by means of φ as follows:

$$T^*(x, y) = \varphi^{(-1)}(\varphi(x) \cdot \varphi(y)),$$

where $\varphi^{(-1)} : [0, 1] \rightarrow [0, 1]$ is a so-called pseudo-inverse of φ . If strictly increasing function φ is left continuous, but non-continuous, then associativity of induces operator is violated, [8]. However, then the operation

$$T_*(x, y) = \begin{cases} \min(x, y) & \text{if } \max(x, y) = 1, \\ \varphi(\varphi^{(-1)}(x) \cdot \varphi^{(-1)}(y)) & \text{otherwise,} \end{cases}$$

defines a non-continuous t-norm. Some example will be given.

1. Introduction

Definition 1. A *triangular norm* (*t-norm* for short) is a binary operation on the unit interval $[0, 1]$, i.e., a function $T : [0, 1]^2 \rightarrow [0, 1]$ such that for all $x, y, z \in [0, 1]$ the following four axioms are satisfied:

(T1) *Commutativity*

$$T(x, y) = T(y, x),$$

(T2) *Associativity*

$$T(x, T(y, z)) = T(T(x, y), z),$$

(T3) *Monotonicity*

$$T(x, y) \leq T(x, z) \quad \text{whenever } y \leq z,$$

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(T4) *Boundary Condition*

$$T(x, 1) = x.$$

Multiplicative generator φ of a triangular norm is a strictly increasing function $\varphi : [0, 1] \rightarrow [0, 1]$ such that $\varphi(1) = 1$ and $\varphi(x) \cdot \varphi(y) \in H(\varphi)$ or $\varphi(x) \cdot \varphi(y) < \varphi(0)$. The corresponding t-norm T is defined by means of φ as follows:

$$T(x, y) = \varphi^{(-1)}(\varphi(x) \cdot \varphi(y)),$$

where $\varphi^{(-1)} : [0, 1] \rightarrow [0, 1]$ is a so-called pseudo-inverse of φ defined by

$$\varphi^{(-1)}(t) = \sup\{x \in [0, 1]; \varphi(x) < t\}$$

with convention $\sup \emptyset = 0$.

Continuous t-norms which are not Archimedean cannot be generated by means of multiplicative (additive) generator. However, there are several non-continuous t-norms which are generated [2], e.g. the drastic product T_W [2], [5]. Assuming the left-continuity of a t-norm, note that only generated t-norms are then continuous and consequently Archimedean [10]. On the other hand, there are examples of non-continuous generated t-norms which are non-Archimedean (and then necessarily not continuous), see [10].

For $x \in]0, 1]$, we can write

$$x = \sum_{i=1}^{\infty} \frac{x_i}{2^i},$$

which is the unique infinite dyadic expansion of x , where $x_i \in \{0, 1\}$ for $i \in \mathbb{N}$. The set $\{i; x_i = 1\}$ is infinite. It is easy to see that each $x \in]0, 1]$ is in a one to one correspondence with $(x_i)_{i \in \mathbb{N}}$, where $x_i \in \{0, 1\}$ and $\text{card } \{i; x_i = 1\}$ is infinite. We will use the following notation:

$$x \approx (x_i)_{i \in \mathbb{N}}.$$

Remark 1. Let $x \approx (x_i)_{i \in \mathbb{N}}$ and $y \approx (y_i)_{i \in \mathbb{N}}$. Then $x < y$ if and only if there exists $k \in \mathbb{N}$ such that for all $i \in \mathbb{N}$, $i \leq k$, we have $x_i = y_i$ and $x_{k+1} < y_{k+1}$. \square

We discuss a t-norm based on above described dyadic expansion, which is generated by non-continuous multiplicative generator.

2. New t-norm

Proposition 1. Let function $g : [0, 1] \rightarrow [0, 1]$ be given by

$$g(x) = \begin{cases} 0 & \text{if } x = 0, \\ \sum_{i=1}^{\infty} \frac{2 \cdot x_i}{3^i} & \text{otherwise,} \end{cases}$$

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where if $x \in]0, 1]$ then $x \approx (x_i)_{i \in N}$. Function g is a strictly increasing and left-continuous.

Proof. Let $x \approx (x_i)_{i \in N}$ and $y \approx (y_i)_{i \in N}$ and $x < y$. Then there exists $k \in N$ such that for $i \in N, i \leq k$, we have $x_i = y_i$ and $x_{k+1} < y_{k+1}$.

Now it is easy to see that $\sum_{i=1}^{\infty} \frac{2 \cdot x_i}{3^i} < \sum_{i=1}^{\infty} \frac{2 \cdot y_i}{3^i}$ and it implies $g(x) < g(y)$.

If $x = 0$ then $g(x) < g(y)$ for $y \in]0, 1]$. It means g is strictly increasing function.

Let $x_0 \in]0, 1]$, $x_0 \approx (a_i)_{i \in N}$ and let $(x^{(n)})_{n \in N}$ be a strictly increasing sequence of reals from $[0, 1]$ such that

$$\lim_{n \rightarrow \infty} x^{(n)} = x_0 \quad \text{and} \quad x^{(n)} \approx (x_i^{(n)})_{i \in N}.$$

Evidently $x^{(n)} < x_0$ for all $n \in N$. From Remark 1 we have

$$0 < x_0 - x^{(n)} < \frac{1}{2^{k_n}},$$

and

$$\lim_{n \rightarrow \infty} x^{(n)} = x_0 \quad \text{implies} \quad \lim_{n \rightarrow \infty} k_n = \infty.$$

Now for any $n \in N$

$$0 < g(x_0) - g(x^{(n)}) = \sum_{i=k_n+1}^{\infty} \frac{2 \cdot (a_i - x_i^{(n)})}{3^i} \leq \frac{2}{3^k}$$

Then

$$\lim_{n \rightarrow \infty} (g(x_0) - g(x^{(n)})) = 0.$$

It means g is left-continuous function. \square

Proposition 2. Each finite dyadic rational is point of discontinuity of function g .

Proof. Let $x_0 = \frac{1}{2^{m_1}} + \frac{1}{2^{m_2}} + \dots + \frac{1}{2^{m_k}}$, $x_0 \approx (a_i)_{i \in N}$ and let x be any real number such that $x > x_0$, $x \in]0, 1]$ and $x \approx (x_i)_{i \in N}$.

Then $a_i = 1$ for $i \geq m_k + 1$ and there exists $i \in N, i < m_k + 1$ that $a_i = 0$ and $x_i = 1$.

Now

$$0 < g(x) - g(x_0) \geq \frac{1}{3^{m_k}},$$

which is violating the right-continuity. \square

Remark 2. We can define $f(x) = g^{(-1)}(x) = \sup\{z \in [0, 1]; g(z) < x\}$ as pseudo-inverse of function g . Because of properties of this function, new function f is continuous and $f^{(-1)} = g$, see [4].

Example 1. Let $T^* : [0, 1]^2 \rightarrow [0, 1]$ be given by

$$T^*(x, y) = f(g(x) \cdot g(y)),$$

where g is a function from Proposition 1 and f is pseudo-inverse of function g . Then T^* is operator, which is not a t-norm. T^* is commutative, monotone, it fulfils boundary condition, however, the associativity is violated. For example

$$T^*\left(\frac{1}{2}, T^*\left(\frac{3}{4}, \frac{3}{4}\right)\right) = \frac{1}{4} < \frac{1}{2} = T^*\left(T^*\left(\frac{1}{2}, \frac{3}{4}\right), \frac{3}{4}\right).$$

On the other hand, the operation

$$T_*(x, y) = \begin{cases} \min(x, y) & \text{if } \max(x, y) = 1, \\ g(f(x) \cdot f(y)) & \text{otherwise,} \end{cases}$$

defines a t-norm. The axioms $T(1)$, $T(3)$, $T(4)$ are evidently fulfilled. Concerning the associativity, it follows from the continuity of f , see [3], [9]. \square

Remark 3. Note that for given functions g and f we have $f(g(x)) = x$ for $x \in]0, 1]$. Therefore $T_*(x, T_*(x, x)) = g(f(x)^3)$ for $x \in]0, 1[$, which implies $x_{T_*}^{(n)} = g(f(x)^n)$ for $x \in]0, 1[$.

3. Some properties of the new t-norm

Now, we will turn our attention to important properties of this t-norm.

(P1) First, we recall the well-known characterization of *left-continuous* t-norms.

Proposition 3. A t-norm T is left-continuous if and only if it is left-continuous in its

first component, i.e., if for each $y \in [0, 1]$ and for each non - decreasing sequence $(x_n)_{n \in \mathbb{N}} \in [0, 1]^{\mathbb{N}}$ we have

$$\sup_{n \in \mathbb{N}} T(x_n, y) = T(\sup_{n \in \mathbb{N}} x_n, y). \quad \square$$

Proposition 4.

- (i) The t-norm T_* is left-continuous on $[0, 1]^2$.
- (ii) The t-norm T_* is continuous in point $(1, 1)$.
- (iii) The t-norm T_* is not left-continuous on $[0, 1]^2$.

Proof.

- (i) The t-norm T_* is defined by $T_*(x, y) = g(f(x) \cdot f(y))$ for $(x, y) \in [0, 1]^2$. The continuity of function f implies the continuity of $f(x) \cdot f(y)$. Function g is left-continuous and consequently the composition $g(f(x) \cdot f(y))$ is left-continuous on $[0, 1]^2$.
- (ii) Let $(x^{(n)})_{n \in \mathbb{N}}$ be a strictly increasing sequence of reals from $[0, 1]$ such that

$$\lim_{n \rightarrow \infty} x^{(n)} = 1 \quad \text{and} \quad x^{(n)} \approx (x_i^{(n)})_{i \in \mathbb{N}}.$$

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Evidently $x^{(n)} < 1$ for all $n \in N$. From Remark 1 we have

$$0 < 1 - x^{(n)} < \frac{1}{2^{k_n}},$$

and

$$\lim_{n \rightarrow \infty} x^{(n)} = 1 \quad \text{implies} \quad \lim_{n \rightarrow \infty} k_n = \infty.$$

Now for any $n \in N$

$$0 < T_*(1, 1) - g(f(x^{(n)}), f(1)) \leq \frac{2}{3^{k_n}}.$$

Then

$$\sup T_*(x^{(n)}, 1) = T_*(\sup x^{(n)}, 1)$$

proving the continuity of T_* in point $(1, 1)$.

- (iii) By definition of T_* we have $T_*(x, 1) = x$ for $x \in [0, 1]$ and $g(f(x), f(1)) = g(f(x))$. But for example: if $x \in]\frac{1}{3}, \frac{2}{3}]$ then $f(x) = \frac{1}{2}$ and $g(f(x), f(1)) = \frac{1}{3} \neq x = T_*(x, 1)$, which is violation of the left-continuity of T_* . \square

- (P2) Now, we recall another definition of *Archimedean* t-norms from [1], [4] which is equivalent with the classical one.

Proposition 5. *A t-norm T is Archimedean if and only if for each $x \in]0, 1[$ we have*

$$\lim_{n \rightarrow \infty} x_T^{(n)} = 0. \quad \square$$

Proposition 6. *The t-norm T_* is Archimedean.*

Proof. Let $x \in]0, 1[$. From Remark 3 we have $x_{T_*}^{(n)} = g(f(x)^n)$ and $f(x) < 1$ for $x \in]0, 1[$.

Therefore

$$\lim_{n \rightarrow \infty} x_{T_*}^{(n)} = \lim_{n \rightarrow \infty} g(f(x)^n) = 0.$$

By Proposition 5, T_* is Archimedean. \square

- (P3) Following important algebraic property is *strict monotonicity*.

Proposition 7. *A t-norm T is strictly monotone if and only if the cancelation law holds, i.e., if $T(x, y) = T(x, z)$ and $x > 0$ imply $y = z$. \square*

Proposition 8. *The t-norm T_* is not strictly monotone.*

Proof. For example $T_*(\frac{1}{2}, \frac{2}{3}) = g(\frac{1}{2} \cdot \frac{1}{2}) = T(\frac{1}{2}, \frac{1}{2})$, which is violation of the strict monotonicity of T_* . \square

(P4)

Definition 2. Let T be a t-norm. An element $a \in]0, 1[$ is called a nilpotent element of T if there exists some $n \in N$ such that $a_T^{(n)} = 0$. \square

The t-norm T_* is an example of t-norm which is left-continuous on $[0, 1]^2$ and continuous in point $(1, 1)$, but non-left-continuous on $[0, 1]^2$. More, this t-norm is Archimedean but neither strictly monotone nor nilpotent in any point from $]0, 1[$.

Indeed, let $a \in]\frac{1}{3^n}, \frac{2}{3^n}]$ for some $n \in N$. Then $a_{T_*}^{(m)} = \frac{1}{3^{m \cdot n}} > 0$ for all $m \in N$. Consequently, no element $b \in]0, 1[$, $b > a$, can be a nilpotent element of T_* , showing the non-existence of any nilpotent element of T_* .

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