

A note on recent results on the law of large numbers for fuzzy numbers *

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ABSTRACT: The law of large numbers for fuzzy numbers is discussed. Several recently published results are either improved, or shown to be false. Some sufficient conditions for specific t-norms are investigated.

Keywords: addition of fuzzy numbers, fuzzy number, law of large numbers, triangular norm

1. Introduction

Fullér [1] proved a law of large numbers for sequences of symmetric triangular fuzzy numbers with common spread. Further investigations were done by Triesch [17], Hong [2,3,4], Hong and Kim [5], Jang and Kwon [6], Salakhutdinov [15], Salakhutdinov and Salakhutdinov [16]. Several results contained in above-mentioned papers can be immediately improved. Some of them are false. And some of them are formulated in a general form, which should be examined in more details for specific t-norms used as a basis for addition of fuzzy numbers.

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Recall first the fuzzy version of the law of large numbers, which concerns the limit properties of T-based sums of sequences of LR-fuzzy numbers, where T is some given triangular norm; see e.g. [7].

Definition 1.

A fuzzy quantity A is a so called *LR-fuzzy number* $A = (a, \alpha, \beta)_{LR}$ if the corresponding membership function satisfies for all $x \in \mathbb{R}$

$$A(x) = \begin{cases} L(\frac{a-x}{\alpha}), & \text{for } a - \alpha \leq x \leq a, \\ R(\frac{x-a}{\beta}), & \text{for } a \leq x \leq a + \beta, \\ 0, & \text{else} \end{cases},$$

where a is the peak of A ; $\alpha > 0$ and $\beta > 0$ is *the left* and *the right spread*, respectively, and L and R are decreasing continuous functions from $[0,1]$ to $[0,1]$ such that $L(0) = R(0) = 1$ and $L(1) = R(1) = 0$. Recall that L and R is called *the left* and *the right shape function*, respectively.

The addition of fuzzy quantities is based on a given t-norm \mathbf{T} , following Zadeh's extension principle, by

$$A \oplus_{\mathbf{T}} B(z) = \sup_{x+y=z} (\mathbf{T}(A(x), B(y))), z \in \mathbb{R}$$

where A, B are given fuzzy quantities. If \mathbf{T} is an Archimedean continuous t-norm with additive generator f then the addition of fuzzy quantities can be expressed as follows

$$A \oplus_{\mathbf{T}} B(z) = f^{(-1)}(\inf_{x+y=z} (f \circ A(x) + f \circ B(y))), \quad z \in \mathbb{R},$$

where $f^{(-1)}$ is the pseudoinverse of f , defined by $f^{(-1)}(x) = f^{-1}(\min(f(0), x))$, $x \in [0, +\infty[$.

Definition 2.

Let $(A_n)_{n \in \mathbb{N}}$, $A_n = (a_n, \alpha_n, \beta_n)_{L_n R_n}$, $n \in \mathbb{N}$, be a sequence of LR-fuzzy numbers (not necessarily with the same shapes and spreads). We say that the sequence $(A_n)_{n \in \mathbb{N}}$

obeys the \mathbf{T} -law of large numbers if $a = \lim_{n \rightarrow \infty} \frac{a_1 + \dots + a_n}{n}$ exists and for all $z \in \mathbb{R}$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} (A_1 \oplus_{\mathbf{T}} \dots \oplus_{\mathbf{T}} A_n)(z) = \chi_a(z),$$

where $\frac{1}{n}A(z) = A(nz)$, $n \in \mathbb{N}$.

The above Definition 2 is a direct generalization of the original Fullér's [1] definition of the \mathbf{T} -law of large numbers. However, its philosophy is not sound. The sum of involved fuzzy numbers is based on some given t-norm \mathbf{T} while the averaging by n corresponds to the min-based addition of fuzzy numbers. Namely, put $\frac{1}{n}A = B_i$, $i = 1, \dots, n$. Then (if A was not crisp) $B_1 \oplus_{\mathbf{T}} \dots \oplus_{\mathbf{T}} B_n = A$ if and only if $\mathbf{T} \text{Ran } A = \min$. As an immediate consequence we have the violating of the \mathbf{T} -law of large numbers for fuzzy numbers whenever \mathbf{T} possesses some non-trivial idempotents, see e.g. counter example of Hwang [4]. More details can be found in Marková-Stupňanová [11].

2. Improvements and corrections of published results

Let \mathbf{T} be a continuous Archimedean t-norm. The results on \mathbf{T} -law of large numbers of Jang and Kwon [6] obtained in their Theorem 1 can be immediately generalized not requiring the concavity of shape function g_n^* from [6]. It is enough to apply the approach of Hong [2] used in his Theorem 1.

On the other hand, Hong's Theorem 1 can be generalized even for sequences of fuzzy numbers without uniformly bounded supports, requiring the next property only:

$$\lim_{n \rightarrow \infty} n f^*(A^{(n)}(x)) \geq f(0) \quad \text{for all } x \neq 0, \quad (\diamond)$$

where f^* is some convex lower bound of an additive generator f of \mathbf{T} , and $A^{(n)}$ is the lowest concave upper bound of centralized fuzzy numbers A_1, \dots, A_n we are dealing with i.e.,

$$A^{(n)}(x) \geq A_i(x - a_i), \forall x \in \mathbb{R}, \forall i \in \{1, \dots, n\}.$$

Several improvements of negative example of Hong [3,4] are contained in Marková-Stupňanová [11].

Example of Hong [4] violating the law of large numbers based on the Hamacher t-norm shows also that the Theorem 1 of Jang and Kwon [6] is not correct. Namely, they should require stronger conditions on the peaks of their fuzzy numbers, e.g. fixed peak for all involved fuzzy numbers.

More serious mistakes are contained in the paper of Salakhutdinov [15] Namely, his Theorem 2 is completely false, see e.g. the counter example of Hong [3]. The main error is hidden in non-correct application of limit properties, claiming that the validity of the T_n -law of large numbers for t-norms $T_1 < \dots < T_n < \dots < T$ with $T = \lim_{n \rightarrow \infty} T_n$ ensures the validity of the T^* -law of large numbers for any $T^* < T$. Similarly, the results of Salakhutdinov and Salakhutdinov [16] continues the above erroneous approach. It is evident from the results of Hwang [2] and Marková-Stupňanová [11] that their hypothesis about the validity of the T -law of large numbers only for not T -stable shapes is not true.

3. Sufficient conditions for the T -law of large numbers in specific cases

Let $A_n = (a, \alpha_n, \beta_n)_{LR}$. Let T be an Archimedean t-norm with an additive generator f . Then the validity of the T -law of large numbers for $(A_n)_{n \in \mathbb{N}}$ is equivalent with the validity of the condition (\diamond) . In the specific case $T = T_P$ (product t-norm) and $L(x) = R(x) = 1 - x$ (linear shapes), $\alpha_n = \beta_n = \sqrt{n}$, T_P -law of large numbers was shown to be true, see [6]. We can improve their example as follows:

Proposition 1.

The sequence $(A_n)_{n \in \mathbb{N}}$, $A_n = (a, \alpha_n, \beta_n)_{LR}$, $n \in \mathbb{N}$, $L(x) = R(x) = 1 - x$, $x \in [0, 1]$, obeys the T_P -law of large numbers whenever $\lim_{n \rightarrow \infty} \frac{n}{c_n} = \infty$, where $c_n = \max(\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n)$.

Proof.

It is immediate that $A^{(n)} = \left(0, \underbrace{\max(\alpha_1, \dots, \alpha_n)}_{a_n}, \underbrace{\max(\beta_1, \dots, \beta_n)}_{b_n} \right)_{LR}$. Recall that

$f(x) = -\log x$ is a (convex) additive generator of t-norm T_P . Then the condition (\diamond) is

fulfilled whenever $\infty = \lim_{n \rightarrow \infty} -n \log \left(L \left(\frac{x}{a_n} \right) \right) = \lim_{n \rightarrow \infty} -n \log \left(R \left(\frac{x}{b_n} \right) \right)$, for any $x > 0$, i.e.,

$$\lim_{n \rightarrow \infty} -n \log \left(1 - \frac{x}{c_n} \right) = +\infty.$$

If $\{c_n\}_{n \in \mathbb{N}}$ is bounded, then the latest equality is obvious (and then $\lim_{n \rightarrow \infty} \frac{n}{c_n} = \infty$). For

$$c_n \rightarrow \infty, \text{ it is } \lim_{n \rightarrow \infty} -n \log \left(1 - \frac{x}{c_n} \right) = x \lim_{n \rightarrow \infty} \frac{n}{c_n} = +\infty \text{ (for all } x > 0) \text{ iff } \lim_{n \rightarrow \infty} \frac{n}{c_n} = \infty.$$

Note that we can generalize Proposition 1 in several directions. Firstly, let f^* be a convex lower bound of an additive generator f of a t-norm \mathbf{T} (if f is itself convex, we can take $f^* = f$). Then the conclusions of Proposition 1 with respect to the \mathbf{T} -law of large numbers are valid whenever $(f^*)'(1^-) < 0$. Recall that in the case of $\mathbf{T} = \mathbf{T}_p$, $(-\log 1^-)' = -1$. For the Hamacher t-norm \mathbf{T}_H , $f(x) = \frac{1}{x} - 1$ and $f'(1^-) = -1$.

For the Yager t-norm \mathbf{T}_p^Y , $p \in]0, \infty[$, $f_p(x) = (1-x)^p$. In the case $p \leq 1$, f_p are concave and the corresponding f^* is, e.g., $f^* = 1-x$ with $(f^*)'(1^-) = -1$. In all these case the Proposition 1 can be applied. However, for $p > 1$, $f_p'(1^-) = 0$, and then the Proposition 1 cannot be applied.

Proposition 2.

The sequence of linear (triangular) fuzzy numbers $(A_n)_{n \in \mathbb{N}}$, $A_n = (a, \alpha_n, \beta_n)$, $n \in \mathbb{N}$ obeys the \mathbf{T}_p^Y -law of large numbers for a given $p > 1$ whenever $\lim_{n \rightarrow \infty} \frac{n}{c_n^p} = \infty$, where c_n is defined as in Proposition 1.

Proof.

It is enough to deal with $\lim_{n \rightarrow \infty} n \left(1 - \left(1 - \frac{x}{c_n} \right) \right)^p = x \lim_{n \rightarrow \infty} \frac{n}{c_n^p}$, $x > 0$. Then

$$x \lim_{n \rightarrow \infty} \frac{n}{c_n^p} \geq 1 = f_p(0) \text{ for all } x > 0 \text{ is equivalent with } \lim_{n \rightarrow \infty} \frac{n}{c_n^p} = \infty.$$

Note that the Proposition 2 can be derived directly from results of Kolesárová [8,9]. Indeed, following [8,9], it is

$$\frac{1}{n} \left(A_1 \oplus_{T_p^Y} \dots \oplus_{T_p^Y} A_n \right) = \left(a, \frac{1}{n} \left(\sum_{i=1}^n \alpha_i^q \right)^{\frac{1}{q}}, \frac{1}{n} \left(\sum_{i=1}^n \beta_i^q \right)^{\frac{1}{q}} \right), \text{ where } \frac{1}{p} + \frac{1}{q} = 1. \text{ Then the } T_p^Y -$$

law of large numbers is equivalent with the equalities

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left(\sum_{i=1}^n \alpha_i^q \right)^{\frac{1}{q}} = \lim_{n \rightarrow \infty} \frac{1}{n} \left(\sum_{i=1}^n \beta_i^q \right)^{\frac{1}{q}} = 0. \quad (\diamond\diamond)$$

It is a matter of simple calculations to show that equalities $(\diamond\diamond)$ are fulfilled whenever

$$\lim_{n \rightarrow \infty} \frac{\alpha_n^p}{n} = \lim_{n \rightarrow \infty} \frac{\beta_n^p}{n} = 0, \text{ and hence when, } \lim_{n \rightarrow \infty} \frac{n}{c_n^p} = \infty.$$

On the other hand, we can generalize also the shapes L and R . If L^* and R^* are the corresponding concave upper bounds, the non-zero value of derivatives $(f^* \circ L^*(0^+))'$ and $(f^* \circ R^*(0^+))'$ allows to apply Proposition 1.

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