ON t-NORM–BASED MULTIPLICATION OF POSITIVE FUZZY INTERVALS

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ABSTRACT. In the paper a general method for multiplying positive fuzzy intervals based on triangular norms is shown. This method can also be used for multiplying positive L-R fuzzy intervals with different shape functions L, R as well as for multiplying positive fuzzy intervals with unbounded supports.

1. INTRODUCTION

In the paper we will deal with triangular norm-based products of positive fuzzy intervals. All known results for the products of L-R fuzzy intervals based on the strongest triangular norm T_M and the weakest T_W, given in [1], [5] and [6] can be obtained by the presented approach. Our results also cover the results for the products based on continuous Archimedean triangular norms given in [2].

We only recall the definitions of positive fuzzy intervals. For the definition and the properties of triangular norms (t-norms for short) which will be used throughout the paper we refer the reader to [4].

Definition 1. A fuzzy subset A of the real line $\mathbb{R}$ is said to be a fuzzy interval if its membership function is continuous and for each $\alpha \in]0,1]$ the corresponding α-cut $A^{(\alpha)} = \{x \in \mathbb{R};\ A(x) \geq \alpha\}$ is a bounded interval in $\mathbb{R}$.

Due to the continuity of the membership function all α-cuts are closed intervals in $\mathbb{R}$, $A^{(\alpha)} = [l_A^{(\alpha)}, r_A^{(\alpha)}] \subset \mathbb{R}$. The kernel of A, i.e., 1-cut $A^{(1)}$, will be denoted by $Ker A = [l_A, r_A]$.

Definition 2. A fuzzy interval will be called positive if $A^{(\alpha)} = [l_A^{(\alpha)}, r_A^{(\alpha)}] \subset]0, \infty[$ for each $\alpha \in]0,1]$.

If A is a positive fuzzy interval then supp $A \subset]0, \infty[$. Since $A(x) = 0$ for each $x \leq 0$, we restrict our considerations to the interval $]0, \infty[$. Note that the membership function is non-decreasing on the interval $]0, l_A]$ and non-increasing on the interval $[r_A, \infty[$.

Key words and phrases. Triangular norm, fuzzy interval, product of fuzzy intervals.
The research was supported by grants MŠ SR 1/4297/97 and Action Cost 15.

2. Product shapes of positive fuzzy intervals

Consider positive fuzzy intervals A, B. The product of A and B based on a t-norm T is a positive fuzzy interval with the membership function given by

$$A \underset{T}{\otimes} B(z) = \sup_{\substack{xy=z \\ x>0, y>0}} T(A(x), B(y)), \quad z > 0. \tag{1}$$

Since $Ker(A \underset{T}{\otimes} B) = KerA.KerB = [l_A l_B, r_A r_B]$ for each t-norm T, for determining $A \underset{T}{\otimes} B$ it is enough to determine the values $A \underset{T}{\otimes} B(z)$ only for $z \in]0, l_A l_B[$ and $z \in]r_A r_B, \infty[$.

For $z \in [r_A r_B, \infty[$ the membership function of the T-product is given by

$$A \underset{T}{\otimes} B(z) = \sup_{\substack{xy=z \\ x \geq r_A, y \geq r_B}} T(A(x), B(y)), \tag{2}$$

which means that the values $A \underset{T}{\otimes} B(z)$ for $z \geq r_A r_B$ depend only on the values $A(x)$ for $x \geq r_A$ and $B(y)$ for $y \geq r_B$. Let us prove the validity of this assertion.

Consider $z \geq r_A r_B$. Let $\epsilon > 0$ such that $r_A - \epsilon > 0$. Then $\frac{z}{r_A - \epsilon} > \frac{z}{r_A} \geq r_B$, which implies $B\left(\frac{z}{r_A - \epsilon}\right) \leq B\left(\frac{z}{r_A}\right)$. This inequality and the properties of t-norms give :

$$T\left(A(r_A - \epsilon), B\left(\frac{z}{r_A - \epsilon}\right)\right) \leq B\left(\frac{z}{r_A - \epsilon}\right) \leq B\left(\frac{z}{r_A}\right) = T\left(A(r_A), B\left(\frac{z}{r_A}\right)\right).$$

Analogouosly, for $\epsilon > 0$ such that $r_B - \epsilon > 0$, we have

$$T\left(A\left(\frac{z}{r_B - \epsilon}\right), B(r_B - \epsilon)\right) \leq A\left(\frac{z}{r_B - \epsilon}\right) \leq A\left(\frac{z}{r_B}\right) = T\left(A\left(\frac{z}{r_B}\right), B(r_B)\right).$$

This means that (2) is valid for each $z \geq r_A r_B$. An analogouos assertion holds for $A \underset{T}{\otimes} B(z)$, $z \in]0, l_A l_B]$.

Consider again $z \geq r_A r_B$ and let $z = xy$, where $x \geq r_A$, $y \geq r_B$. The elements x and y can be expressed in the form $x = \frac{r_A}{u}$, $y = \frac{r_B}{v}$, where $u, v \in]0, 1]$. Therefore $z = \frac{r_A r_B}{t}$, $t = uv$, $u, v \in]0, 1]$.
Using (2), for $z \geq r_A r_B$ we obtain

$$A \underset{T}{\otimes} B(z) = \sup_{\substack{uv=t \\ u,v \in]0,1]}} T\left(A\left(\frac{r_A}{u}\right), B\left(\frac{r_B}{v}\right)\right) \quad , \quad t = \frac{r_A r_B}{z}.$$

Put

$$A^*(u) = A\left(\frac{r_A}{u}\right) \quad , \quad B^*(v) = B\left(\frac{r_B}{v}\right). \tag{3}$$

Then

$$A \underset{T}{\otimes} B(z) = \sup_{\substack{uv=t \\ u,v\in]0,1]}} T(A^*(u), B^*(v)) \;, \quad t = \frac{r_A r_B}{z}. \tag{4}$$

The functions A^*, B^* are defined on the interval $]0,1]$. They are non-decreasing, continuous and have a strict maximum $A^*(1) = 1$ and $B^*(1) = 1$.

Similarly, if $z \in]0, l_A l_B]$ such that $z = xy$, where $x \in]0, l_A]$ and $y \in]0, l_B]$, then $x = l_A u$, $y = l_B v$, $u, v \in]0,1]$, and we can write $z = l_A l_B t$, $t = uv$, $u, v \in]0,1]$. For the T-product $A \underset{T}{\otimes} B$ we obtain

$$A \underset{T}{\otimes} B(z) = \sup_{\substack{uv=t \\ u,v\in]0,1]}} T(A(l_A u), B(l_B v)) = \sup_{\substack{uv=t \\ u,v\in]0,1]}} T(A_*(u), B_*(v)) \;, \quad t = \frac{z}{l_A l_B}, \tag{5}$$

where

$$A_*(u) = A(l_A u) \quad \text{and} \quad B_*(v) = B(l_B v). \tag{6}$$

The functions A_*, B_* are defined on the interval $]0,1]$ and have the same properties as the functions A^*, B^*.

Let $\mathbb{S}$ be a set of all functions $S :]0,1] \to]0,1]$ which are non-decreasing, continuous, $S(0+) = \lim_{x\to 0+} S(x) = 0$ and $S(x) = 1$ iff $x = 1$. A function $S \in \mathbb{S}$ will be called a shape.
The full name is a product shape to distinguish shapes of this type from shapes introduced in [6].

According to (3) and (6), it holds :

$$A(x) = \begin{cases} A_*\left(\frac{x}{l_A}\right) & x \in]0, l_A] \\ 1 & x \in [l_A, r_A] \\ A^*\left(\frac{r_A}{x}\right) & x \in [r_A, \infty[, \end{cases} \tag{7}$$

where $A_*, A^* \in \mathbb{S}$.

There is a one-to-one correspondence between positive fuzzy intervals and their shapes and kernels. A positive fuzzy interval A is determined by its kernel $Ker A = [l_A, r_A]$ and shapes A_*, A^*. Therefore a positive fuzzy interval will be denoted by $A = (l_A, r_A, A_*, A^*)$.

Example 1. Let A be a positive fuzzy interval whose membership function is given by

$$A(x) = \begin{cases} \frac{x}{2} & x \in]0,2] \\ 1 & x \in [2,3] \\ 4 - x & x \in [3,4] \\ 0 & x \in [4, \infty[. \end{cases}$$

Then $A = (2, 3, A_*, A^*)$, where

$$A_*(t) = A(2t) = t \;, \; t \in]0,1],$$

$$A^*(t) = A\left(\frac{3}{t}\right) = \max(0, 4 - \frac{3}{t}) = \begin{cases} 0 & t \in]0, \frac{3}{4}] \\ 4 - \frac{3}{t} & t \in [\frac{3}{4}, 1]. \end{cases}$$

Example 2. Let
$$A(x) = \begin{cases} 0 & x \in]0,1] \\ x-1 & x \in [1,2] \\ \frac{2}{x} & x \in [2,\infty[. \end{cases}$$
Then $A = (2, 2, A_*, A^*)$, where
$$A_*(t) = A(2t) = \max(2t-1, 0) = \begin{cases} 0 & t \in]0, 0.5] \\ 2t-1 & t \in [0.5, 1], \end{cases}$$
$$A^*(t) = A\left(\frac{2}{t}\right) = t \ , \ t \in]0,1].$$

Example 3. Let
$$A(x) = \begin{cases} \exp(1-\frac{1}{x}) & x \in]0,1] \\ \exp(1-x) & x \in [1,\infty[. \end{cases}$$
Then $A = (1, 1, A_*, A^*)$, where
$$A_*(t) = A^*(t) = \exp(1 - \frac{1}{t}) \ , \ t \in]0,1].$$

For a given t-norm T, the T-product of shapes $S_1, S_2 \in \mathbb{S}$ is due to (4) and (5) defined by
$$S_1 \underset{T}{\otimes} S_2(t) = \sup_{\substack{uv=t \\ u,v \in]0,1]}} T(S_1(u), S_2(v)) \ , \quad t \in]0,1].$$
Stress that unlike the T-products of positive fuzzy intervals, the T-products of shapes are defined on the interval $]0,1]$.

The shape $(A \underset{T}{\otimes} B)^*$ is by (7) defined as a function satisfying
$$(A \underset{T}{\otimes} B)(z) = (A \underset{T}{\otimes} B)^* \left(\frac{r_A r_B}{z}\right), \quad z \in [r_A r_B, \infty[.$$
However, by (4), for $(A \underset{T}{\otimes} B)(z)$ it is also true that
$$(A \underset{T}{\otimes} B)(z) = (A^* \underset{T}{\otimes} B^*) \left(\frac{r_A r_B}{z}\right), \quad z \in [r_A r_B, \infty[.$$
Moreover, it is evident that $(A \underset{T}{\otimes} B)^*(z) = 1$ iff $z = 1$ and $(A \underset{T}{\otimes} B)^*(0^+) = 0$. Therefore $(A \underset{T}{\otimes} B)^* = A^* \underset{T}{\otimes} B^*$, which means that the shape of a T-product $A \underset{T}{\otimes} B$ is equal to the T-product of shapes of incoming members A, B. An analogouos assertion holds for $(A \underset{T}{\otimes} B)_*$.

Summarizing, the T-product of positive fuzzy intervals A, B is a positive fuzzy interval
$$A \underset{T}{\otimes} B = \left(l_A l_B, r_A r_B, A_* \underset{T}{\otimes} B_*, A^* \underset{T}{\otimes} B^*\right). \qquad (8)$$

3. Products of positive fuzzy intervals based on marginal t-norms

Due to (8), it is enough to characterize T-products of shapes. As previously explained, having an output formula for a T-product of shapes, we can determine the corresponding T-product of given positive fuzzy intervals.

In what follows, we show the application of this method only for the marginal t-norms T_M,T_W. Recall that T_M is the strongest t-norm, and it is defined by $T_M(x,y) = \min(x,y)$, $x, y \in [0,1]$. The weakest t-norm $T_W : [0,1]^2 \to [0,1]$ is given by

$$T_W(x,y) = \begin{cases} \min(x,y) & \text{if } \max(x,y) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Following the ideas in [3], [6], we define the quasi-inverse of a non-decreasing, right continuous function $h :]0,1] \to]0,1]$ as the function

$$h^{(-1)}(s) = \inf\{t \in]0,1];\ h(t) > s\}.$$

The quasi-inverse is a non-decreasing right continuous function and $(h^{(-1)})^{(-1)} = h$. If h is a bijection then $h^{(-1)} = h^{-1}$, where h^{-1} is the inverse function of the function h.

Note that though considered shapes are continuous, their quasi-inverses are in general only right continuous functions.

T_W-products of positive fuzzy intervals.

Proposition 1. *Let S_1, S_2 be shapes. Then*

$$S_1 \underset{T_W}{\otimes} S_2 = \max(S_1, S_2).$$

Proof. Any T-product of the shapes S_1, S_2 can be expressed in the form :

$$S_1 \underset{T}{\otimes} S_2(t) = \sup_{u \in [t,1]} T\left(S_1(u), S_2\left(\frac{t}{u}\right)\right), \quad t \in]0,1].$$

The drastic product T_W has zero values with the exception of the case when either $S_1(u) = 1$ or $S_2\left(\frac{t}{u}\right) = 1$. $S_1(u) = 1$ iff $u = 1$ and $S_2\left(\frac{t}{u}\right) = 1$ iff $u = t$, consequently,

$$S_1 \underset{T}{\otimes} S_2(t) = \max(S_1(t), S_2(t)).$$

Example 4. Let A, B be positive fuzzy intervals,

$$A(x) = \begin{cases} 2x - x^2 & x \in]0,2] \\ 0 & x \in [2,\infty[\end{cases} \quad \text{and} \quad B(x) = \begin{cases} 0 & x \in]0,\frac{1}{2}] \\ \frac{1}{2}(2x-1) & x \in [\frac{1}{2},\frac{3}{2}] \\ \frac{3}{2x} & x \in [\frac{3}{2},\infty[. \end{cases}$$

Then $A = (1, 1, A_*, A^*)$, where $A_*(t) = 2t - t^2$, $A^*(t) = \max(0, \frac{2}{t} - \frac{1}{t^2})$, $t \in]0, 1]$ and $B = (\frac{3}{2}, \frac{3}{2}, B_*, B^*)$, whereby $B_*(t) = \max(0, \frac{1}{2}(3t-1))$ and $B^*(t) = t$, $t \in]0, 1]$. Applying Proposition 1, we obtain $A_* \underset{T_W}{\otimes} B_* = A_*$ and $A^* \underset{T_W}{\otimes} B^* = B^*$. This means that

$$A \underset{T_W}{\otimes} B = (\frac{3}{2}, \frac{3}{2}, A_*, B^*) \text{ , i.e., } A \underset{T_W}{\otimes} B(z) = \begin{cases} \frac{4}{3}z - \frac{4}{9}z^2 & z \in]0, \frac{3}{2}] \\ \frac{3}{2z} & z \in [\frac{3}{2}, \infty[. \end{cases}$$

***T_M*-products of positive fuzzy intervals.**

T_M-products of shapes can be easily determined by their quasi-inverses.

Proposition 2. *Let S_1, S_2 be shapes. Then*

$$\left(S_1 \underset{T_M}{\otimes} S_2\right)^{(-1)} = S_1^{(-1)}.S_2^{(-1)} . \tag{9}$$

We omit the details of the proof.

Since ${}^{(-1)} : S \mapsto S^{(-1)}$ is an involutive operation, (9) can be expressed in the form

$$(S_1 \underset{T_M}{\otimes} S_2) = \left(S_1^{(-1)}.S_2^{(-1)}\right)^{(-1)} . \tag{10}$$

Example 5. Let A be a positive fuzzy interval given in Ex.1 and B be a positive fuzzy interval with the membership function

$$B(x) = \min(1, \max(0, 2 - |x - 3|)) .$$

We determine the T_M-product of A and B .

By Ex.1, $A = (2, 3, P, Q)$ with $P(t) = t$ and $Q(t) = \max(0, 4 - \frac{3}{t})$, $t \in]0, 1]$. The fuzzy interval B can be written in the form $B = (2, 4, R, S)$, where

$$R(t) = B_*(t) = B(2t) = \max(0, 2t - 1) = \begin{cases} 0 & t \in]0, 0.5] \\ 2t - 1 & t \in [0.5, 1] \end{cases}$$

and

$$S(t) = B^*(t) = B\left(\frac{4}{t}\right) = \max(0, 5 - \frac{4}{t}) = \begin{cases} 0 & t \in]0, 0.8] \\ 5 - \frac{4}{t} & t \in [0.8, 1] \end{cases} .$$

Denote $C = A \underset{T_M}{\otimes} B$. Then $C = (4, 12, C_*, C^*)$ and the shapes C_*, C^* can be determined by (10). Since $P^{(-1)}(s) = P^{-1}(s) = s$, $R^{(-1)}(s) = \frac{s+1}{2}$, for $D_*^{(-1)}$ and its quasi-inverse we obtain

$$D_*^{(-1)}(s) = \frac{s^2 + s}{2} \quad , \quad D_*(t) = -\frac{1}{2} + \frac{1}{2}\sqrt{1 + 8t}, \ t \in]0, 1].$$

Next,

$$Q^{(-1)}(s) = \frac{3}{4-s} \text{ and } S^{(-1)}(s) = \frac{4}{5-s}, \ s \in]0,1].$$

Therefore $(D^*)^{(-1)}(s) = \frac{12}{(4-s)(5-s)}$, and the quasi-inverse is given by

$$D^*(t) = \max\left(0, \frac{9}{2} - \frac{1}{2}\sqrt{1+\frac{48}{t}}\right) = \begin{cases} 0 & t \in]0, 0.6] \\ \frac{9}{2} - \frac{1}{2}\sqrt{1+\frac{48}{t}} & t \in [0.6, 1]. \end{cases}$$

Applying (7), we obtain the resulting formula for the T_M-product D :

$$D(z) = \begin{cases} -\frac{1}{2} + \frac{1}{2}\sqrt{1+2z} & z \in]0,4] \\ 1 & z \in [4,12] \\ \frac{9}{2} - \frac{1}{2}\sqrt{1+4z} & z \in [12,20] \\ 0 & z \in [20, \infty[. \end{cases}$$

The fuzzy intervals A, B considered in the previous example, are special types of so-called *L-R* fuzzy intervals with linear shape functions $L(x) = R(x) = 1 - x$, $x \in [0,1]$. In the same way as in Ex.5 we could derive the resulting formula for the T_M-product of any positive *L-R* fuzzy intervals and obtain the results which coincide with the known results given in [1], [5], [7].

We only show the advantages of the presented method for multiplying positive fuzzy intervals, namely, the possibility of multiplying positive *L-R* fuzzy intervals with different shape functions L, R and fuzzy intervals with unbounded supports.

Example 6. Let A be a positive fuzzy interval given in Ex.3. We have shown that $A = (1,1,P,P)$, where $P(t) = \exp(1 - \frac{1}{t})$, $t \in]0,1]$. Let us denote $A \underset{T_M}{\otimes} A = (1,1,Q,Q)$. Since $P^{(-1)}(s) = P^{-1}(s) = \frac{1}{1-\log s}$, $s \in]0,1]$, using (10), for the shape Q, we obtain $Q(t) = \exp(1 - \frac{1}{\sqrt{t}})$, $t \in]0,1]$. Therefore, by (7)

$$A \underset{T_M}{\otimes} A(z) = \begin{cases} \exp(1 - \frac{1}{\sqrt{z}}) & z \in]0,1] \\ \exp(1 - \sqrt{z}) & z \in [1, \infty[. \end{cases}$$

Example 7. Let A, B be positive fuzzy intervals ,

$$A(x) = \begin{cases} \frac{x}{2} & x \in]0,2] \\ \frac{2}{x} & x \in [2, \infty[, \end{cases} \qquad B(x) = \begin{cases} 2x - x^2 & x \in]0,1] \\ \frac{1}{x} & x \in [1, \infty[. \end{cases}$$

This means that

$$A = (2,2,P,Q), \text{ where } P(t) = Q(t) = t, t \in]0,1]$$

and

$$B = (1,1,R,S) \text{ , where } R(t) = 2t - t^2, \ S(t) = t, \ t \in]0,1].$$

Denote $C = A \underset{T_M}{\otimes} B$. Since $P^{(-1)}(s) = s$ and $R^{(-1)}(s) = 1 - \sqrt{1-s}$, the problem of determining C_* becomes the problem of solving the equation $s(1 - \sqrt{1-s}) = t$. It can be shown that the solution having all required properties is

$$s = \frac{2}{3}\sqrt{6}\sqrt{t}\cos\left(\frac{1}{3}\arcsin\left(\frac{3\sqrt{6}\sqrt{t}}{8}\right) + \frac{\pi}{6}\right),$$

and therefore $C_*(t)$ is equal to the right hand side of the previous expression for any $t \in]0,1]$.
Since $Q^{(-1)}.S^{(-1)}(s) = s^2$, we have $C^*(t) = \sqrt{t}, t \in]0,1]$. Applying (7), we obtain the resulting formula for the T_M-product of A and B :

$$C(z) = \begin{cases} \frac{2}{\sqrt{3}}\sqrt{z}\cos\left(\frac{1}{3}\arcsin\left(\frac{3\sqrt{3}}{8}\sqrt{z}\right) + \frac{\pi}{6}\right) & z \in]0,2] \\ \frac{\sqrt{2}}{\sqrt{z}} & z \in [2, \infty[. \end{cases}$$

4. Conclusions

We have presented a general method for multiplying positive fuzzy intervals. As we have shown there is a one-to-one correspondence between positive fuzzy intervals and their kernels and product shapes. The multiplying positive fuzzy intervals can be reduced to the multiplying corresponding product shapes. Several examples are illustrating possibility of using this method.

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