B-B Fuzzy Sets $\underset{\sim}{S}$ (III)*

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Abstract

This paper proposes the theory of B-B fuzzy set which is the continuation of [1], [2].

In the study of O-B fuzzy set theory (L.A.Zadeh fuzzy set theory), due to the introduction of λ-cutset S_λ , there generate union-ordinary resolution theorem [3], intersection-ordinary resolution theorem [4] of O-B fuzzy set.

[1], [2] propose B-B fuzzy set $\underset{\sim}{S}$. Due to the introduction of λ-cutset S_λ, there generate union-ordinary resolution theorem, intersection-ordinary resolution theorem of B-B fuzzy set $\underset{\sim}{S}$.

Naturally people propose such questions : B-B fuzzy set can be resolved into many ordinary sets S_λ ; can B-B fuzzy set $\underset{\sim}{S}$ be resolved into many fuzzy sets $\underset{\sim}{S}^\alpha$? This paper explains that B-B fuzzy set $\underset{\sim}{S}$ can be resolved into many fuzzy sets $\underset{\sim}{S}^\alpha$. (O-B fuzzy set $\underset{\sim}{A}$ can also be resolved into many fuzzy sets $\underset{\sim}{A}^\alpha$, which has not been studied in O-B fuzzy set theory)

This paper proposes the concept of α-imbedding set $\underset{\sim}{S}^\alpha$ of $\underset{\sim}{S}$, α-imbedding theorem of $\underset{\sim}{S}$, α-imbedding fuzzy resolution theorem of $\underset{\sim}{S}$.

Keywords : $\underset{\sim}{S}$ set, α-imbedding set $\underset{\sim}{S}^\alpha$, α-imbedding theorem, α-imbedding union-fuzzy resolution theorem, α-imbedding intersection-fuzzy resolution theorem, the relational theorem of α-imbedding union-fuzzy resolution and α-imbedding intersection-fuzzy resolution .

1. Introduction

People proved that O-B fuzzy set $\underset{\sim}{A}$ could be resolved into many ordinary sets A_λ (union-ordinary resolution [3], intersection-ordinary resolution [4]) in O-B fuzzy set theory. [2] proved that B-B fuzzy set $\underset{\sim}{S}$ could be resolved into many ordinary sets S_λ (union-ordinary resolution, intersection-ordinary resolution). The above successful study all has the aid of the concept of λ-cutset.

In the concept of λ-cutset, there hides a very important and useful concept which has not been found and applied presently. This concept is said to be α-imbedding set $\underset{\sim}{S}^\alpha$ of $\underset{\sim}{S}$ in this paper. With the help of

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the concept of α-imbedding set $\underset{\sim}{S}^{\alpha}$, this paper proves that B-B fuzzy set $\underset{\sim}{S}$ can be resolved into many fuzzy sets $\underset{\sim}{S}^{\alpha}$, meanwhile, this paper also gives that O-B fuzzy set $\underset{\sim}{A}$ can be resolved into many fuzzy sets $\underset{\sim}{A}^{\alpha}$, which has not been studied presently.

In the study of B-B fuzzy control, people propose such a question: can B-B fuzzy set $\underset{\sim}{S}$ be resolved into many α-imbedding sets $\underset{\sim}{S}^{\alpha} \in \mathscr{F}(X)$? (B-B fuzzy control and its control reasoning, control program, control algorithm will be studied in later thesises, which are omitted here); so we must make clear if B-B fuzzy set $\underset{\sim}{S}$ can be resolved into many fuzzy sets $\underset{\sim}{S}^{\alpha}$ and some relational results.

When we review the present fuzzy control study, we can find that all the study is carried out on the basis of O-B fuzzy set theory (L.A.Zadeh fuzzy set theory). In fact, these fuzzy control study is a integer-semi regional fuzzy control. The control program and algorithm provided by the integer-semi regional fuzzy control are elementary, and the fuzzy control problems solved by integer-semi regional fuzzy control are very limit, which are as limited as those problems solved in integer range. We come up against difficulties when we continue to study the present integer-semi regional fuzzy control, just as people can not express such concepts as $\frac{1}{2}$, $\frac{2}{3}$, $\frac{7}{9}$, ... in integer range.

The above simple analysis is the background of the study of this paper.

Appointed in the following that $\underset{\sim}{S}$, $\underset{\sim}{S}^{\alpha}$ are normal fuzzy sets in X ; $\exists x_0 \in X^{+} \cup X^{\circ}$, s.t. $\underset{\sim}{S}(x_0)=\bigvee_{i=1}^{m}\underset{\sim}{S}(x_i)$, $\underset{\sim}{S}^{\alpha}(x_0)=\bigvee_{i=1}^{m}\underset{\sim}{S}^{\alpha}(x_i)$; $\exists x_0' \in X^{-} \cup X^{\circ}$, s.t. $\underset{\sim}{S}(x_0')=\bigwedge_{i=1}^{m}\underset{\sim}{S}(x_i)$, $\underset{\sim}{S}^{\alpha}(x_0')=\bigwedge_{i=1}^{m}\underset{\sim}{S}^{\alpha}(x_i)$.

For avoiding confusion, $\alpha_1 \in [0, 1]$ and $\alpha_2 \in [-1, 0]$ with $|\alpha_1| = |\alpha_2|$ are both expressed by $\alpha \in [-1, 1]$.

2. α-Imbedding Set $\underset{\sim}{S}^{\alpha}$ of $\underset{\sim}{S}$

Definition 2.1 Let S_λ be λ-cutset of B-B fuzzy set $\underset{\sim}{S}$ [1], for any $\alpha \in [-1, 1]$, if $|\alpha| \le |\lambda|$, then $\underset{\sim}{S}^{\alpha}$ is said to be a α-full imbedding set of $\underset{\sim}{S}$ generated by S_λ:

$$\underset{\sim}{S}^{\alpha}: X \to [-1, 1]$$

$$x \to \underset{\sim}{S}^{\alpha}(x) \tag{2.1}$$

$\underset{\sim}{S}^{\alpha}(x)$ is said to be the imbedding fuzzy kiss function of $x \in X$ concerning $\underset{\sim}{S}^{\alpha}$, for simplicity is said to be the fuzzy kiss function; for given $x_0 \in X$, $\underset{\sim}{S}^{\alpha}(x_0)$ is said to be the fuzzy kiss measure of x_0 concerning $\underset{\sim}{S}^{\alpha}$.

The number α is said to be the imbedding measure of $\underset{\sim}{S}^{\alpha}$ concerning $\underset{\sim}{S}$, $-1<\alpha<1$.

From definition 2.1 we get

1°. $\exists x_0 \in X^+ \cup X^\circ$, s.t. $0 < \underset{\sim}{S}^\alpha(x_0) = \alpha = \lambda$, $\lambda \in (0, 1)$.

2°. $\exists x_0' \in X^- \cup X^\circ$, s.t. $0 > \underset{\sim}{S}^\alpha(x_0') = \alpha = \lambda$, $\lambda \in (-1, 0)$.

Definition 2.2 Let S_λ be λ-cutset of B-B fuzzy set $\underset{\sim}{S}$, for any $\alpha \in [-1, 1]$, if $|\alpha| < |\lambda|$, then $\dot{\underset{\sim}{S}}^\alpha$ is said to be a α-lacking imbedding set.

Clearly, $x \in X^+ \cup X^\circ$, $\dot{\underset{\sim}{S}}^\alpha(x) < \lambda$; $x \in X^- \cup X^\circ$, $-\dot{\underset{\sim}{S}}^\alpha(x) > -\lambda$.

Theorem 2.1 Let $\underset{\sim}{S}^\alpha$ be a α-full imbedding set of B-B fuzzy set $\underset{\sim}{S}$, then there exist unique $x_0 \in X^+ \cup X^\circ$, $x_0' \in X^- \cup X^\circ$ which satisfy

$$\Big| \underset{x_0 \in X^+ \cup X^\circ}{\underset{\sim}{S}^\alpha(x_0)} \Big| = \Big| \underset{x_0' \in X^- \cup X^\circ}{\underset{\sim}{S}^\alpha(x_0')} \Big| = |\pm\lambda| \tag{2.2}$$

In fact, any α-imbedding set $\underset{\sim}{S}^\alpha$ is a normal fuzzy set ; Clearly there exists $x_0 \in X^+ \cup X^\circ$ so that

$$\underset{\sim}{S}^\alpha(x_0) = \bigvee_{i=1}^{m} \underset{\sim}{S}^\alpha(x_i) = \lambda$$

there exists $x_0' \in X^- \cup X^\circ$ so that

$$\underset{\sim}{S}^\alpha(x_0') = \bigwedge_{i=1}^{m} \underset{\sim}{S}^\alpha(x_i) = -\lambda$$

Theorem 2.2 There exist many α-imbedding sets $\underset{\sim}{S}^{\alpha_i}$ in B-B fuzzy set $\underset{\sim}{S}$ so that the relation of $\underset{\sim}{S}^{\alpha_i}$ and $\underset{\sim}{S}$ satisfies

$$1^\circ.\quad \bigcup_{i=1}^{m} \underset{\sim}{S}^{\alpha_i} = \underset{\sim}{S}^+, \quad \underset{\sim}{S}^{\alpha_i}, \ \underset{\sim}{S}^+ \in \mathscr{F}(X^+ \cup X^\circ)$$

$$2^\circ.\quad \bigcap_{i=1}^{m} \underset{\sim}{S}^{\alpha_i} = \underset{\sim}{S}^-, \quad \underset{\sim}{S}^{\alpha_i}, \ \underset{\sim}{S}^- \in \mathscr{F}(X^- \cup X^\circ) \tag{2.3}$$

Here, $\underset{\sim}{S}^+$ is up-branch set of $\underset{\sim}{S}$, $\underset{\sim}{S}^-$ is down-branch set of $\underset{\sim}{S}$.

Corollary There exist many α-imbedding sets $\underset{\sim}{A}^{\alpha_i}$ in O-B fuzzy set $\underset{\sim}{A}$ so that the relation of $\underset{\sim}{A}^{\alpha_i}$ and $\underset{\sim}{A}$ satisfies

$$\bigcup_{\substack{i=1 \\ \alpha_i \in [0,1]}}^{m} \underset{\sim}{A}^{\alpha_i} = \underset{\sim}{A} \tag{2.4}$$

Theorem 2.2 and its corollary are direct results, which proof are omitted.

Theorem 2.3 $\underset{\sim}{S}^\alpha$ is a unique α-full imbedding set of B-B fuzzy set $\underset{\sim}{S}$ if and only if the imbedding measure α of $\underset{\sim}{S}^\alpha$ satisfies

$$\alpha = |\pm 1| \tag{2.5}$$

3. α-Imbedding Union-Fuzzy Resolution of $\underset{\sim}{S}$

Theorem 3.1 Let $\underset{\sim}{S}^{\alpha}$ be the α-full imbedding set of $\underset{\sim}{S}$, $\alpha \in [-1, 1]$; then

$$\underset{\sim}{S} = \bigcup_{\alpha \in [-1,1]} \alpha \underset{\sim}{S}^{\alpha} \tag{3.1}$$

Proof : For any $x \in X$, we only need prove (where $X = X^{+} \cup X^{-} \cup X^{\circ}$)

$$\underset{\sim}{S}(x) = (\bigcup_{\alpha \in [-1,1]} \alpha \underset{\sim}{S}^{\alpha})(x)$$

1°. Let $\alpha \in [0, 1]$, for any $x \in X^{+} \cup X^{\circ}$, $(\bigcup_{\alpha \in [0,1]} \alpha \underset{\sim}{S}^{\alpha})(x) = \bigvee_{\alpha \in [0,1]} (\alpha \wedge \underset{\sim}{S}^{\alpha}(x)) = \bigvee_{\alpha \in [0,1]} (y | \underset{\sim}{S}^{\alpha}(x) \leq \alpha)$

$= \bigvee_{\forall \lambda < \alpha \in [0,1]} (y | S_{\lambda}^{\alpha}(x) = 1) = \bigvee_{\forall \lambda < \alpha \in [0,1]} (y | \underset{\sim}{S}^{\alpha}(x) \geq \lambda) = \underset{\sim}{S}(x)$

hence

$$\underset{\sim}{S} = \bigcup_{\alpha \in [0,1]} \alpha \underset{\sim}{S}^{\alpha} \tag{3.2}$$

2°. Let $\alpha \in [-1, 0]$, for any $x \in X^{-} \cup X^{\circ}$, $(\bigcup_{\alpha \in [-1,0]} \alpha \underset{\sim}{S}^{\alpha})(x) = \bigvee_{\alpha \in [-1,0]} (\alpha \wedge \underset{\sim}{S}^{\alpha}(x)) = \bigvee_{\alpha \in [-1,0]} (y | \underset{\sim}{S}^{\alpha}(x) \geq \alpha)$

$= \bigvee_{\forall \lambda > \alpha \in [-1,0]} (y | S_{\lambda}^{\alpha}(x) = -1) = \bigvee_{\forall \lambda > \alpha \in [-1,0]} (y | \underset{\sim}{S}^{\alpha}(x) \leq \lambda) = \underset{\sim}{S}(x)$

hence

$$\underset{\sim}{S} = \bigcup_{\alpha \in [-1,0]} \alpha \underset{\sim}{S}^{\alpha} \tag{3.3}$$

Due to (3.2), (3.3) we have

$$\underset{\sim}{S} = \bigcup_{\alpha \in [-1,1]} \alpha \underset{\sim}{S}^{\alpha} \tag{3.4}$$

Here we point out that

1. (2.5) of [2] gave

$$\underset{\sim}{S} = \bigcup_{\alpha \in [-1,1]} \alpha S_{\lambda}^{\alpha} \tag{3.5}$$

In (3.5) $\alpha S_{\lambda}^{\alpha} \in \mathscr{P}(X)$, so (3.5) expresses the relation of fuzzy set $\underset{\sim}{S}$ and ordinary set $\alpha S_{\lambda}^{\alpha}$. In (3.4) $\alpha \underset{\sim}{S}^{\alpha} \in \mathscr{F}(X)$, so (3.4) expresses the relation of fuzzy set $\underset{\sim}{S}$ and fuzzy set $\alpha \underset{\sim}{S}^{\alpha}$.

2. (3.4), (3.5) are both union-resolution forms of fuzzy set $\underset{\sim}{S}$, which are very similar in form. In fact, (3.4), (3.5) express two different concepts without any relation. (3.5) is the fuzzy-ordinary union-resolution of $\underset{\sim}{S}$, and it expresses that fuzzy set $\underset{\sim}{S}$ can be composed of the union of a string of ordinary sets $\alpha S_{\lambda}^{\alpha}$; (3.4) is the fuzzy-fuzzy union-resolution of $\underset{\sim}{S}$, and it expresses that fuzzy set $\underset{\sim}{S}$ can be composed of the union of a string of fuzzy sets $\alpha \underset{\sim}{S}^{\alpha}$.

Due to theorem 3.1 we can get

Theorem 3.2 Let $\underset{\sim}{\dot{S}}^{\alpha}$ be the α-lacking imbedding set of $\underset{\sim}{S}$, $\alpha \in [-1, 1]$; then

$$\underset{\sim}{S} = \bigcup_{\alpha \in [-1,1]} \alpha \underset{\sim}{\dot{S}}^{\alpha} \tag{3.6}$$

The proof is similar to that of theorem 3.1, which is omitted here.

Theorem 3.3 Let $\underset{\sim}{S}^{\alpha}$ be the α-imbedding set of $\underset{\sim}{S}$, $\underset{\sim}{S}^{\alpha} \in \mathscr{F}(X)$; Let

$$\underset{\sim}{H}^{\alpha}: \quad [-1, 1] \rightarrow \mathscr{F}(X)$$

$$\alpha \rightarrow \underset{\sim}{H}^{\alpha}(\alpha)$$

satisfy $\dot{\underset{\sim}{S}}^{\alpha} \subseteq \underset{\sim}{H}^{\alpha}(\alpha) \subseteq \underset{\sim}{S}^{\alpha}$, for any $\alpha \in [-1, 1]$, then

$$\underset{\sim}{S} = \bigcup_{\alpha \in [-1,1]} \alpha \underset{\sim}{H}^{\alpha}(\alpha) \tag{3.7}$$

Proof: 1°. Let $\alpha \in [-1, 0]$ and $\dot{\underset{\sim}{S}}^{\alpha} \subseteq \underset{\sim}{H}^{\alpha}(\alpha) \subseteq \underset{\sim}{S}^{\alpha}$, then $\alpha \dot{\underset{\sim}{S}}^{\alpha} \subseteq \alpha \underset{\sim}{H}^{\alpha}(\alpha) \subseteq \alpha \underset{\sim}{S}^{\alpha}$; therefore,

$$\underset{\sim}{S} = \bigcup_{\alpha \in [-1,0]} \alpha \dot{\underset{\sim}{S}}^{\alpha} \subseteq \bigcup_{\alpha \in [-1,0]} \alpha \underset{\sim}{H}^{\alpha}(\alpha) \subseteq \bigcup_{\alpha \in [-1,0]} \alpha \underset{\sim}{S}^{\alpha} = \underset{\sim}{S}$$

i.e.

$$\underset{\sim}{S} = \bigcup_{\alpha \in [-1,0]} \alpha \underset{\sim}{H}^{\alpha}(\alpha) \tag{3.8}$$

2°. Let $\alpha \in [0, 1]$ and $\dot{\underset{\sim}{S}}^{\alpha} \subseteq \underset{\sim}{H}^{\alpha}(\alpha) \subseteq \underset{\sim}{S}^{\alpha}$, then $\alpha \dot{\underset{\sim}{S}}^{\alpha} \subseteq \alpha \underset{\sim}{H}^{\alpha}(\alpha) \subseteq \alpha \underset{\sim}{S}^{\alpha}$; therefore,

$$\underset{\sim}{S} = \bigcup_{\alpha \in [0,1]} \alpha \dot{\underset{\sim}{S}}^{\alpha} \subseteq \bigcup_{\alpha \in [0,1]} \alpha \underset{\sim}{H}^{\alpha}(\alpha) \subseteq \bigcup_{\alpha \in [0,1]} \alpha \underset{\sim}{S}^{\alpha} = \underset{\sim}{S}$$

i.e.

$$\underset{\sim}{S} = \bigcup_{\alpha \in [0,1]} \alpha \underset{\sim}{H}^{\alpha}(\alpha) \tag{3.9}$$

Due to (3.8), (3.9), we get (3.7):

$$\underset{\sim}{S} = \bigcup_{\alpha \in [-1,1]} \alpha \underset{\sim}{H}^{\alpha}(\alpha) \tag{3.10}$$

From theorem 3.1, 3.2, 3.3, we get directly

Theorem 3.4 Let $\underset{\sim}{S}^{\alpha}$ be α-imbedding set of $\underset{\sim}{S}$, $\underset{\sim}{S}^{\alpha} \in \mathscr{F}(X)$; if $\alpha_1 < \alpha_2$, $\alpha_1, \alpha_2 \in [-1, 1]$, then

$$\underset{\sim}{H}^{\alpha}(\alpha_1) \subset \underset{\sim}{H}^{\alpha}(\alpha_2) \tag{3.11}$$

Theorem 3.5 Let $\underset{\sim}{S}^{\alpha}$ be α-imbedding set of $\underset{\sim}{S}$, $\underset{\sim}{S}^{\alpha} \in \mathscr{F}(X)$, $\alpha \in [-1, 1]$, then

1°.
$$\dot{\underset{\sim}{S}}^{\alpha} \subseteq \bigcap_{\alpha' < \alpha} \underset{\sim}{H}^{\alpha}(\alpha') \tag{3.12}$$

2°.
$$\underset{\sim}{S}^{\alpha} \supseteq \bigcup_{\alpha' < \alpha} \underset{\sim}{H}^{\alpha}(\alpha') \tag{3.13}$$

Here we point out that :

if [-1, 1] turns into [0, 1], B-B fuzzy set $\underset{\sim}{S}$ of this paper degenerates into O-B fuzzy set $\underset{\sim}{A}$, and theorem 3,1, 3.2, 3.3, 3.4, 3.5 degenerate into α-union fuzzy resolutions of L.A.Zadeh fuzzy set theory, theorem 3,1, 3.2, 3.3 turn into three α-imbedding union-fuzzy resolution theorems of L.A.Zadeh fuzzy set theory, i.e.

Theorem 3.6 Let $\underset{\sim}{A}^{\alpha}$ be α-full imbedding set of $\underset{\sim}{A}$, $\alpha \in [0, 1]$; then

$$\underset{\sim}{A} = \bigcup_{\alpha \in [0,1]} \alpha \underset{\sim}{A}^{\alpha} \tag{3.14}$$

Theorem 3.7 Let $\underset{\sim}{A}^{\alpha}$ be α-lacking imbedding set of $\underset{\sim}{A}$, $\alpha \in [0, 1]$; then

$$\underset{\sim}{A} = \bigcup_{\alpha \in [0,1]} \alpha \dot{\underset{\sim}{A}}^{\alpha} \tag{3.15}$$

Theorem 3.8 Let $\underset{\sim}{A}^{\alpha}$ be α-imbedding set of $\underset{\sim}{A}$, $\alpha \in [0, 1]$,

$$\underset{\sim}{H}^{\alpha}: \quad [0, 1] \to \mathscr{F}(X)$$

$$\alpha \to \underset{\sim}{H}^{\alpha}(\alpha)$$

satisfy $\dot{\underset{\sim}{A}}^{\alpha} \subseteq \underset{\sim}{H}^{\alpha}(\alpha) \subseteq \underset{\sim}{A}^{\alpha}$, then

$$\underset{\sim}{A} = \bigcup_{\alpha \in [0,1]} \alpha \underset{\sim}{H}^{\alpha}(\alpha) \tag{3.16}$$

4. α-Imbedding Intersection-Fuzzy Resolution of $\underset{\sim}{S}$

Definition 4.1 Let $\underset{\sim}{S}^{\alpha}$ be α-imbedding of $\underset{\sim}{S}$, $\alpha \in [-1, 1]$; provided that $\alpha \underset{\sim}{S}^{\alpha} \in \mathscr{F}(X)$, fuzzy kiss function of $\alpha \underset{\sim}{S}^{\alpha}$ is

$$(\alpha \underset{\sim}{S}^{\alpha})(x) = \alpha \vee \underset{\sim}{S}^{\alpha}(x) \tag{4.1}$$

Theorem 4.1 Let $\underset{\sim}{S}^{\alpha}$ be α-full imbedding set of $\underset{\sim}{S}$, $\alpha \in [-1, 1]$; then

$$\underset{\sim}{S} = \bigcap_{\alpha \in [-1,1]} \alpha \underset{\sim}{S}^{\alpha} \tag{4.2}$$

Proof: 1°. Let $\alpha \in [0, 1]$, $\forall x \in X^{+} \cup X^{\circ}$, $(\bigcap_{\alpha \in [0,1]} \alpha \underset{\sim}{S}^{\alpha})(x) = \bigwedge_{\alpha \in [0,1]} (\alpha \underset{\sim}{S}^{\alpha})(x) = \bigwedge_{\alpha \in [0,1]} (\alpha \vee \underset{\sim}{S}^{\alpha}(x))$

$$= \bigwedge_{\alpha \in [0,1]} \underset{\sim}{S}^{\alpha}(x) = \bigwedge_{\alpha<\lambda<1} \{ y | S^{\alpha}_{\lambda}(x) = 1 \} = \bigwedge_{\substack{\alpha<\lambda<1 \\ \forall\alpha, \forall\lambda}} \{ y | \underset{\sim}{S}^{\alpha}(x) \geq \lambda \} = \underset{\sim}{S}(x)$$

hence
$$\underset{\sim}{S} = \bigcap_{\alpha \in [0,1]} \alpha \underset{\sim}{S}^{\alpha} \tag{4.3}$$

2°. Let $\alpha \in [-1, 0]$, $\forall x \in X^{-} \cup X^{\circ}$, $(\bigcap_{\alpha \in [-1,0]} \alpha \underset{\sim}{S}^{\alpha})(x) = \bigwedge_{\alpha \in [-1,0]} (\alpha \underset{\sim}{S}^{\alpha})(x) = \bigwedge_{\alpha \in [-1,0]} (\alpha \vee \underset{\sim}{S}^{\alpha}(x))$

$$= \bigwedge_{\alpha \in [-1,0]} \underset{\sim}{S}^{\alpha}(x) = \bigwedge_{-1<\lambda<\alpha} \{ y | S^{\alpha}_{\lambda}(x) = -1 \} = \bigwedge_{\substack{-1<\lambda<\alpha \\ \forall\alpha, \forall\lambda}} \{ y | \underset{\sim}{S}^{\alpha}(x) \leq \lambda \} = \underset{\sim}{S}(x)$$

hence
$$\underset{\sim}{S} = \bigcap_{\alpha \in [-1,0]} \alpha \underset{\sim}{S}^{\alpha} \tag{4.4}$$

Due to (4.3), (4.4) we get (4.2):

$$\underset{\sim}{S} = \bigcap_{\alpha \in [-1,1]} \alpha \underset{\sim}{S}^{\alpha} \tag{4.5}$$

Here we point out :

(3.2) of [2] gave

$$\underset{\sim}{S} = \bigcap_{\alpha \in [-1,1]} \alpha S^{\alpha}_{\lambda} \tag{4.6}$$

(4.6) is fuzzy-ordinary intersection resolution of $\underset{\sim}{S}$, which expresses that fuzzy set $\underset{\sim}{S}$ can be composed of the intersection of a string of ordinary sets $\alpha S_{\lambda}^{\alpha}$; (4.4) is fuzzy-fuzzy intersection resolution of $\underset{\sim}{S}$, which expresses that fuzzy set $\underset{\sim}{S}$ can be composed of the intersection of a string of fuzzy sets $\alpha \underset{\sim}{S}^{\alpha}$. Though (4.4) is similar to (3.2) of [2] in form, (4.4) is essentially different from (3.2) of [2].

From theorem 4.1 we get

Theorem 4.2 Let $\dot{\underset{\sim}{S}}^{\alpha}$ be α-lacking imbedding set of $\underset{\sim}{S}$, $\alpha \in [-1, 1]$; then

$$\underset{\sim}{S} = \bigcap_{\alpha \in [-1,1]} \alpha \dot{\underset{\sim}{S}}^{\alpha} \tag{4.7}$$

The proof of theorem 4.2 is similar to that of theorem 4.1, which is omitted.

Theorem 4.3 Let $\underset{\sim}{S}^{\alpha}$ be α-imbedding set of $\underset{\sim}{S}$, $\alpha \in [-1, 1]$; Let

$$\underset{\sim}{H}^{\alpha} : \quad [-1, 1] \rightarrow \mathscr{F}(X)$$

$$\alpha \rightarrow \underset{\sim}{H}^{\alpha}(\alpha)$$

satisfy $\alpha \in [-1, 1]$, $\dot{\underset{\sim}{S}}^{\alpha} \subseteq \underset{\sim}{H}^{\alpha}(\alpha) \subseteq \underset{\sim}{S}^{\alpha}$, then

$$\underset{\sim}{S} = \bigcap_{\alpha \in [-1,1]} \alpha \underset{\sim}{H}^{\alpha}(\alpha) \tag{4.8}$$

Proof: 1°. Let $\alpha \in [-1, 0]$, $\dot{\underset{\sim}{S}}^{\alpha} \subseteq \underset{\sim}{H}^{\alpha}(\alpha) \subseteq \underset{\sim}{S}^{\alpha}$, then $\alpha \dot{\underset{\sim}{S}}^{\alpha} \subseteq \alpha \underset{\sim}{H}^{\alpha}(\alpha) \subseteq \alpha \underset{\sim}{S}^{\alpha}$, therefore, $\underset{\sim}{S} = \bigcap_{\alpha \in [-1,0]} \alpha \dot{\underset{\sim}{S}}^{\alpha} \subseteq \bigcap_{\alpha \in [-1,0]} \alpha \underset{\sim}{H}^{\alpha}(\alpha) \subseteq \bigcap_{\alpha \in [-1,0]} \alpha \underset{\sim}{S}^{\alpha} = \underset{\sim}{S}$

i.e.

$$\underset{\sim}{S} = \bigcap_{\alpha \in [-1,0]} \alpha \underset{\sim}{H}^{\alpha}(\alpha) \tag{4.9}$$

2°. Let $\alpha \in [0, 1]$, $\dot{\underset{\sim}{S}}^{\alpha} \subseteq \underset{\sim}{H}^{\alpha}(\alpha) \subseteq \underset{\sim}{S}^{\alpha}$, then $\alpha \dot{\underset{\sim}{S}}^{\alpha} \subseteq \alpha \underset{\sim}{H}^{\alpha}(\alpha) \subseteq \alpha \underset{\sim}{S}^{\alpha}$, therefore, $\underset{\sim}{S} = \bigcap_{\alpha \in [0,1]} \alpha \dot{\underset{\sim}{S}}^{\alpha} \subseteq \bigcap_{\alpha \in [0,1]} \alpha \underset{\sim}{H}^{\alpha}(\alpha) \subseteq \bigcap_{\alpha \in [0,1]} \alpha \underset{\sim}{S}^{\alpha} = \underset{\sim}{S}$

i.e.

$$\underset{\sim}{S} = \bigcap_{\alpha \in [0,1]} \alpha \underset{\sim}{H}^{\alpha}(\alpha) \tag{4.10}$$

Due to (4.9), (4.10), we get (4.8):

$$\underset{\sim}{S} = \bigcap_{\alpha \in [-1,1]} \alpha \underset{\sim}{H}^{\alpha}(\alpha) \tag{4.11}$$

From theorem 4.1, 4.2, we get directly

Theorem 4.4 Let $\underset{\sim}{S}^{\alpha}$ be α-imbedding set of $\underset{\sim}{S}$, $Q = [-\alpha', \alpha'] \subseteq [-1, 1]$, then

$$\underset{\sim}{S} = \bigcap_{\alpha \in Q} \alpha \dot{\underset{\sim}{S}}^{\alpha} \tag{4.12}$$

Here we point out

if [-1, 1] turns into [0, 1], B-B fuzzy set $\underset{\sim}{S}$ of this paper degenerates into O-B fuzzy set $\underset{\sim}{A}$; theorem 4.1, 4.2, 4.3, 4.4 degenerate into α-intersection fuzzy resolutions of L.A.Zadeh fuzzy set theory; theorem 4.1, 4.2, 4.3 are three α-imbedding intersection fuzzy resolution theorems of L.A.Zadeh fuzzy set theory, i.e.

Theorem 4.5 Let $\underset{\sim}{A}^{\alpha}$ be α-full imbedding set of $\underset{\sim}{A}$, $\alpha \in [0, 1]$; then

$$\underset{\sim}{A} = \bigcap_{\alpha \in [0,1]} \alpha \underset{\sim}{A}^{\alpha} \tag{4.13}$$

Theorem 4.6 Let $\dot{\underset{\sim}{A}}^{\alpha}$ be α-lacking imbedding set of $\underset{\sim}{A}$, $\alpha \in [0, 1]$; then

$$\underset{\sim}{A} = \bigcap_{\alpha \in [0,1]} \alpha \dot{\underset{\sim}{A}}^{\alpha} \tag{4.14}$$

Theorem 4.7 Let $\underset{\sim}{A}^{\alpha}$ be α-imbedding set of $\underset{\sim}{A}$, $\alpha \in [0, 1]$, and

$$\underset{\sim}{H}^{\alpha}: \quad [0, 1] \rightarrow \mathscr{F}(X)$$
$$\alpha \rightarrow \underset{\sim}{H}^{\alpha}(\alpha)$$

satisfy $\dot{\underset{\sim}{A}}^{\alpha} \subseteq \underset{\sim}{H}^{\alpha}(\alpha) \subseteq \underset{\sim}{A}^{\alpha}$, then

$$\underset{\sim}{A} = \bigcap_{\alpha \in [0,1]} \alpha \underset{\sim}{H}^{\alpha}(\alpha) \tag{4.15}$$

5. Relations of α-Imbedding Union-Fuzzy Resolution − α-Imbedding Intersection-Fuzzy Resolution

Theorem 5.1 Let $\bigcup_{\alpha \in [-1,1]} \alpha \underset{\sim}{S}^{\alpha}$, $\bigcap_{\alpha \in [-1,1]} \alpha \underset{\sim}{S}^{\alpha}$ be α-imbedding union-fuzzy resolution, α-imbedding intersection-fuzzy resolution of $\underset{\sim}{S}$, respectively, then for any $\alpha \in [-1, 1]$,

$$\bigcup_{\alpha \in [-1,1]} \alpha \underset{\sim}{S}^{\alpha} = \bigcap_{\alpha \in [-1,1]} \alpha \underset{\sim}{S}^{\alpha} \tag{5.1}$$

Its proof can be gotten directly by (3.1), (4.2), which is omitted.

Theorem 5.2 Let $\bigcup_{\alpha \in [-1,1]} \alpha \dot{\underset{\sim}{S}}^{\alpha}$, $\bigcap_{\alpha \in [-1,1]} \alpha \dot{\underset{\sim}{S}}^{\alpha}$ be α-imbedding union-fuzzy resolution, α-imbedding intersection-fuzzy resolution of $\underset{\sim}{S}$, respectively, then $\forall \alpha \in [-1, 1]$,

$$\bigcup_{\alpha \in [-1,1]} \alpha \dot{\underset{\sim}{S}}^{\alpha} = \bigcap_{\alpha \in [-1,1]} \alpha \dot{\underset{\sim}{S}}^{\alpha} \tag{5.2}$$

Its proof can be gotten directly by (3.6), (4.7), which is omitted.

Theorem 5.3 Let $\bigcup_{\alpha \in [-1,1]} \alpha \underset{\sim}{H}^{\alpha}(\alpha)$, $\bigcap_{\alpha \in [-1,1]} \alpha \underset{\sim}{H}^{\alpha}(\alpha)$ be α-imbedding union-fuzzy resolution, α-imbedding intersection-fuzzy resolution of $\underset{\sim}{S}$, respectively; then for any $\alpha \in [-1, 1]$,

$$\bigcup_{\alpha \in [-1,1]} \alpha \underset{\sim}{H}^{\alpha}(\alpha) = \bigcap_{\alpha \in [-1,1]} \alpha \underset{\sim}{H}^{\alpha}(\alpha) \tag{5.3}$$

Its proof can be gotten directly by (3.7), (4.8), which is omitted.

Due to above results, we get three relational theorems of α-imbedding union-fuzzy resolution and α-imbedding intersection-fuzzy resolution in L.A.Zadeh fuzzy set theory.

Theorem 5.4 Let $\bigcup_{\alpha\in[0,1]} \alpha \underset{\sim}{A}^{\alpha}$, $\bigcap_{\alpha\in[0,1]} \alpha \underset{\sim}{A}^{\alpha}$ be α-imbedding union-fuzzy resolution of $\underset{\sim}{A}$, α-imbedding intersection-fuzzy resolution of $\underset{\sim}{A}$, respectively; then for any $\alpha\in[0, 1]$,

$$\bigcup_{\alpha\in[0,1]} \alpha \underset{\sim}{A}^{\alpha} = \bigcap_{\alpha\in[0,1]} \alpha \underset{\sim}{A}^{\alpha} \tag{5.4}$$

Its proof can be gotten directly by (3.14), (4.13), which is omitted.

Theorem 5.5 Let $\bigcup_{\alpha\in[0,1]} \alpha \underset{\sim}{\dot{A}}^{\alpha}$, $\bigcap_{\alpha\in[0,1]} \alpha \underset{\sim}{\dot{A}}^{\alpha}$ be α-imbedding union-fuzzy resolution of $\underset{\sim}{A}$, α-imbedding intersection-fuzzy resolution of $\underset{\sim}{A}$, respectively; then for any $\alpha\in[0, 1]$,

$$\bigcup_{\alpha\in[0,1]} \alpha \underset{\sim}{\dot{A}}^{\alpha} = \bigcap_{\alpha\in[0,1]} \alpha \underset{\sim}{\dot{A}}^{\alpha} \tag{5.5}$$

Its proof can be gotten directly by (3.15), (4.14), which is omitted.

Theorem 5.6 Let $\bigcup_{\alpha\in[0,1]} \alpha \underset{\sim}{H}^{\alpha}(\alpha)$, $\bigcap_{\alpha\in[0,1]} \alpha \underset{\sim}{H}^{\alpha}(\alpha)$ be α-imbedding union-fuzzy resolution of $\underset{\sim}{A}$, α-imbedding intersection-fuzzy resolution of $\underset{\sim}{A}$, respectively; then for any $\alpha\in[0, 1]$,

$$\bigcup_{\alpha\in[0,1]} \alpha \underset{\sim}{H}^{\alpha}(\alpha) = \bigcap_{\alpha\in[0,1]} \alpha \underset{\sim}{H}^{\alpha}(\alpha) \tag{5.6}$$

Its proof can be gotten directly by (3.16), (4.15), which is omitted.

Due to the analysis of 3, 4, 5 of this paper, we get following propositions:

Proposition 1 There exist dual, equal fuzzy resolution forms in B-B fuzzy set $\underset{\sim}{S}$: α-imbedding union-fuzzy resolution of $\underset{\sim}{S}$, α-imbedding intersection-fuzzy resolution of $\underset{\sim}{S}$.

Proposition 2 There exist dual, equal fuzzy resolution forms in O-B fuzzy set $\underset{\sim}{A}$: α-imbedding union-fuzzy resolution of $\underset{\sim}{A}$, α-imbedding intersection-fuzzy resolution of $\underset{\sim}{A}$.

Proposition 3 Dual equal resolution forms of B-B fuzzy set $\underset{\sim}{S}$ degenerate into corresponding dual, equal resolution forms of O-B fuzzy set $\underset{\sim}{A}$ under certain conditions.

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