B-B Fuzzy Sets $\underset{\approx}{S}$ (I)*

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Abstract

This paper proposes the theory of B-B fuzzy set $\underset{\sim}{S}$ (Both-Branch fuzzy set $\underset{\sim}{S}$).

B-B fuzzy set $\underset{\sim}{S}$ is the following concept : the relation $\underset{\sim}{S}(x)$ of any $x \in X$ and B-B fuzzy set $\underset{\sim}{S}$ defined in X satisfies $\underset{\sim}{S}(x) \in [-1, 1]$. $\underset{\sim}{S}(x)$ is said to be the fuzzy kiss function of element x concerning B-B fuzzy set $\underset{\sim}{S}$. For a given $x_0 \in X$, $\underset{\sim}{S}(x_0)$ is said to be the fuzzy kiss measure of B-B fuzzy set $\underset{\sim}{S}$.

In 1965, outstanding scholar L.A. Zadeh extended the relation of element $x \in X$ and general set $A \in \mathscr{P}(X)$ defined in universe X from $\chi_A^{(x)} \in \{0, 1\}$ to $\mu_{\underset{\sim}{A}}^{(x)} \in [0, 1]$, where $\chi_A^{(x)}$ is a characteristic function, and $\mu_{\underset{\sim}{A}}^{(x)}$ is a membership function ; due to this L.A. Zadeh proposed fuzzy set and its general theory which is a creative and outstanding academic contribution.

In the fuzzy set $\underset{\sim}{A}$ proposed by L.A.Zadeh, the relation $\mu_{\underset{\sim}{A}}^{(x)}$ of $x \in X$ and $\underset{\sim}{A}$ is defined in [0, 1], i.e. the relation $\mu_{\underset{\sim}{A}}^{(x)}$ of all x in X and $\underset{\sim}{A}$ takes positives values. There exist such facts in engineering decisions : $\mu_{\underset{\sim}{A}}^{(x_i)} \in (0, 1]$, $\mu_{\underset{\sim}{A}}^{(x_j)} \in [-1, 0)$, $\mu_{\underset{\sim}{A}}^{(x_k)} = 0$, where x_i, x_j, x_k are some elements in X. The element $x_j \in X$ is very important in fuzzy decisions and fuzzy control, and we can't discard it.

Due to above facts, this paper proposes B-B fuzzy set $\underset{\sim}{S}$. The fuzzy set $\underset{\sim}{A}$ proposed by L.A. Zadeh is said to be O-B fuzzy set (one-branch fuzzy set) in this paper.

Keywords : B-B fuzzy set, Fuzzy kiss function, Character theorem of B-B fuzzy set.

1. Introduction

In 1965, L.A. Zadeh proposed fuzzy set theory, which is said to be O-B fuzzy set (one-branch fuzzy set) in this paper and which has been studied by many scholars of fuzzy set theory and application in thirty years in the world. Many outstanding results have been published. In these academic results, all sholars comply with such a rule : let X be a universe, $\underset{\sim}{A}$ a fuzzy set defined in X, the relation of any

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$x_i \in X$ and $\underset{\sim}{A}$ satisfies mapping :

$$\mu: \quad X \rightarrow [0, 1]$$
$$x \rightarrow \mu_{\underset{\sim}{A}}^{(x)} \qquad (1.1)$$

There exists a common fact in fuzzy decisions of many systems such as : industrial engineering, economy, traffic, medical treatment, environmental protection, earthquake, water conservancy, meteorology. The fact is that let X be a universe, and $\underset{\sim}{S}(x_i) \in (0, 1]$, $\underset{\sim}{S}(x_j) \in [-1, 0)$, $\underset{\sim}{S}(x_k) = 0$, where x_i, x_j, x_k are some elements of X, and $\underset{\sim}{S}(x)$ is the relation of $x \in X$ and $\underset{\sim}{S}$. This is mapping :

$$\underset{\sim}{S}: \quad X \rightarrow [-1, 1]$$
$$x \rightarrow \underset{\sim}{S}(x) \qquad (1.2)$$

Due to above analysis, we give such a universe X : X is composed of X^+, X^-, X°, and

$$X = X^+ \cup X^- \cup X^\circ \qquad (1.3)$$

Here for any $x \in X^+$, $\underset{\sim}{S}(x)$ satisfies in X^+

$$1 \geq \underset{\sim}{S}(x) > 0 \qquad (1.4)$$

for any $x \in X^-$, $\underset{\sim}{S}(x)$ satisfies in X^-

$$0 > \underset{\sim}{S}(x) \geq -1 \qquad (1.5)$$

for any $x \in X^\circ$, $\underset{\sim}{S}(x)$ satisfies in X°

$$\underset{\sim}{S}(x) = 0 \qquad (1.6)$$

So we continue to study the fuzzy set proposed by L.A. Zadeh, in which $\mu_{\underset{\sim}{A}}^{(x)}$ satisfies $\mu_{\underset{\sim}{A}}^{(x)} \in [0, 1]$. Clearly X defined in fuzzy set theory of L.A. Zadeh can be treated as X^+ of this paper.

In fact, for any $x_j \in X^-$, $X^- \subset X$, $-1 \leq \underset{\sim}{S}(x_j) \leq 0$ plays an important role in fuzzy decisions, fuzzy control which can't be neglected.

This paper proposes the concept of B-B fuzzy set $\underset{\sim}{S}$, and studies its general problems which come from following fuzzy decisive problems.

Let X be a set of all factors which have relations with " beneficial trend " $\underset{\sim}{Q}$ in fuzzy evaluation and fuzzy decision system of "beneficial trend " (" beneficial trend" takes profit increasing or decreasing as its object of evaluation generally, and it is also a fuzzy concept). $X = \{ x_1, x_2, \ldots, x_n \}$. The relation factor $x \in X$ and $\underset{\sim}{Q}$ is denoted by fuzzy kiss function $\underset{\sim}{Q}(x)$. Some factors x_α in X take positive effect on the rise of $\underset{\sim}{Q}$, the degree of which can be expressed by fuzzy kiss measure $\underset{\sim}{Q}(x_\alpha) \in (0, 1]$ of fuzzy kiss function ; some factors x_β take reverse effect on the rise of $\underset{\sim}{Q}$, the degree of which can be expressed by fuzzy kiss measure $\underset{\sim}{Q}(x_\beta) \in [-1, 0)$ of fuzzy kiss function ; other factors x_k take " neutral " effect on the rise of $\underset{\sim}{Q}$, the degree of which can be expressed by fuzzy kiss measure $\underset{\sim}{Q}(x_k) = 0$ of fuzzy kiss function. There exist above facts in all engineering decisive systems, even in the disease diagnosing, disease treating.

In fact , the first thing that our predecessors and we ourselves begin to do before they implement decision $\underset{\sim}{Q}$(or draw up a control regulation) is : to take all factors $x_i \in X$ that have relations with decision $\underset{\sim}{Q}$ and have positive, reverse comparison with $\underset{\sim}{Q}$ many times. If " positive decision trend " is larger than " reverse decision trend " (or the obtaining is more than the losing), people implement this decision ; if " positive decision trend " is less than " reverse decision trend " (or the obtaining is less than the losing), people abandon this decision.

Here we point out that

1°. If we don't consider the reverse effect on decision $\underset{\sim}{Q}$ of factors x_j in $X = \{ x_1, x_2, \dots, x_n \}$(or there doesn't exist $x_j \in X$ such that $\underset{\sim}{S}(x_j) \in [-1, 0)$, or $X^- = \varnothing$), $\underset{\sim}{Q}(x_j) \in [-1, 0)$ doesn't exist. Then B-B fuzzy set $\underset{\sim}{S}$ proposed in this paper degenerates into O-B fuzzy set $\underset{\sim}{A}$, i.e. $\underset{\sim}{S} = \underset{\sim}{A}$.

2°. Generally, the relation of all factors x in X and decision $\underset{\sim}{Q}$ doesn't always satisfy $\underset{\sim}{Q}(x) \in [0, 1]$.

3°. Presently, none has introduced the relation $\underset{\sim}{Q} \in [-1, 0)$ into the study of fuzzy decision and fuzzy control.

2. B-B Fuzzy Set $\underset{\sim}{S}$

Appointed that the following defined B-B fuzzy set refers to normal B-B fuzzy set, which for simplicity is said to be B-B fuzzy set.

Definition 2.1 Let X be a universe, X^+, X^-, X° are said to be up-universe, down-universe, bound-universe, respectively, if

1°. for any $x_i \in X^+$, the relation of x_i and $\underset{\sim}{S}$ satisfies

$$1 \geq \underset{\sim}{S}(x_i) > 0 \tag{2.1}$$

2°. for any $x_j \in X^-$, the relation of x_j and $\underset{\sim}{S}$ satisfies

$$0 > \underset{\sim}{S}(x_j) \geq -1 \tag{2.2}$$

3°. for any $x_k \in X^\circ$, the relation of x_k and $\underset{\sim}{S}$ satisfies

$$\underset{\sim}{S}(x_k) = 0 \tag{2.3}$$

From definition 2.1 we have :

1°. $X = X^{+} \cup X^{-} \cup X^{\circ}$

2°. if $X^{-} = \varnothing$, (1.3) turns into

$$X = X^{+} \cup X^{\circ} \qquad (2.4)$$

Here X of (2.4) turns into the universe of fuzzy set defined by L.A. Zadeh .

Definition 2.2 Given mapping

$$\underset{\sim}{S}: \quad X \rightarrow [-1, 1]$$

$$x \rightarrow \underset{\sim}{S}(x) \qquad (2.5)$$

$\underset{\sim}{S}(x)$ defines a B-B fuzzy set $\underset{\sim}{S}$ in X, and $\underset{\sim}{S}(x)$ is said to be fuzzy kiss function of x concerning $\underset{\sim}{S}$. For given $x_0 \in X$, $\underset{\sim}{S}(x_0)$ is said to be fuzzy kiss measure of x_0 concerning $\underset{\sim}{S}$.

B-B fuzzy set $\underset{\sim}{S}$ in finite universe X can be expressed by a fuzzy vector

$$\underset{\sim}{S} = (\underset{\sim}{S}(x_1)/x_1,\ \underset{\sim}{S}(x_2)/x_2,\ \underset{\sim}{S}(x_3)/x_3,\ \underset{\sim}{S}(x_4)/x_4,\ \underset{\sim}{S}(x_5)/x_5)$$

$$= (-0.6 \quad -0.4 \quad 0.3 \quad 1 \quad -0.2) \qquad (2.6)$$

Here $X = \{x_1, x_2, x_3, x_4, x_5\}$, $\underset{\sim}{S} \in \mathscr{F}(X)$.

Both-branch fuzzy set $\underset{\sim}{S}$ in infinite universe X can be expressed by

$$\underset{\sim}{S} = \int_X (\underset{\sim}{S}(x)/x) \qquad (2.7)$$

Proposition 2.1 Let $\underset{\sim}{S}$ be B-B fuzzy set in X , then there exist x_α , $x_\beta \in X$ such that $\underset{\sim}{S}(x_\alpha) = 1$, $\underset{\sim}{S}(x_\beta) = -1$; On the contrary it is also true.

Proposition 2.2 Let $\underset{\sim}{S}$ be B-B fuzzy set in X, if $X^{-} = \varnothing$ and

$$X = X^{+} \cup X^{\circ} \qquad (2.8)$$

then B-B fuzzy set $\underset{\sim}{S}$ degenerates into O-B fuzzy set $\underset{\sim}{A}$ (L.A. Zadeh fuzzy set), fuzzy kiss function $\underset{\sim}{S}(x)$ degenerates into fuzzy membership function $\mu_{\underset{\sim}{A}}^{(x_i)}$, i.e.

$$1°. \qquad \underset{\sim}{S} = \underset{\sim}{A} \qquad (2.9)$$

$$2°. \quad \text{for any } x_i \in X,\ \underset{\sim}{S}(x_i) = \mu_{\underset{\sim}{A}}^{(x_i)} \qquad (2.10)$$

Proposition 2.3 There exists $X^{\circ} = X^{+} \cap X^{-}$ in X, and for all $x_0 \in X^{\circ}$, then

$$\underset{\sim}{S}(x_0) = 0 \qquad (2.11)$$

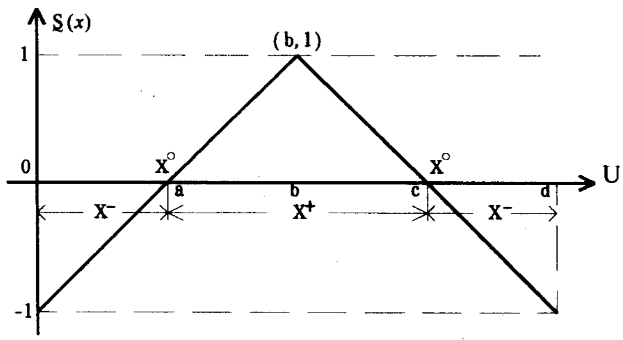
Proposition 2.1, 2.2, 2.3 are clear facts which are omitted.

The existence sense of bound-universe $X^{\circ} = X^{+} \cap X^{-}$ of B-B fuzzy set $\underset{\sim}{S}$:

There always exists such $x_0 \in X$ in fuzzy decision (or in fuzzy control) that $x_0 \in X^{\circ} \subset X$ offsets positive decision relation $\underset{\sim}{S}(x) \in (0, 1]$ and reverse decision relation $\underset{\sim}{S} \in [-1, 0)$ into null, and the relation of x_0 and $\underset{\sim}{S}$ satisfies $\underset{\sim}{S}(x_0) = 0$. It is clear from graph 2.3 that these x_0 constitute X°. In fact, bound-universe X° is the dividing point of $\underset{\sim}{S}(x)$, and X° is also the turning point of fuzzy inference.

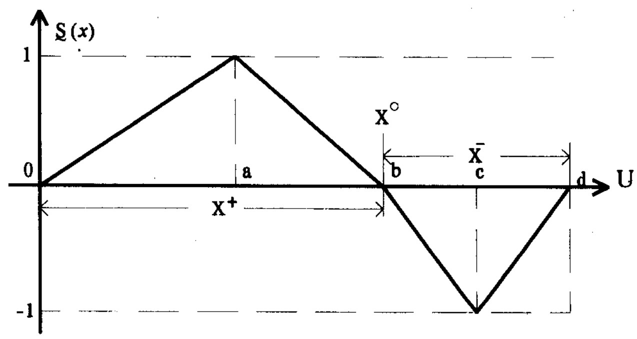
For the later study, we give the graph of B-B fuzzy kiss function $\underset{\sim}{S}(x)$ in finite universe X.

1°. the graph of linear symmetric B-B fuzzy kiss function $\underset{\sim}{S}(x)$ in finite universe X :



Graph 2.1 Linear symmetric fuzzy kiss function $\underset{\sim}{S}(x)$ in X

2°. the graph of linear nonsymmetric B-B fuzzy kiss function $\underset{\sim}{S}(x)$ in finite universe X:



Graph 2.2 Linear nonsymmetric fuzzy kiss function $\underset{\sim}{S}(x)$ in X

The forms of linear symmetric fuzzy kiss function $\underset{\sim}{S}(x)$ and linear nonsymmetric fuzzy kiss function $\underset{\sim}{S}(x)$ in real field $\mathscr{R}$ corresponding to graph 2.1, 2.2, respectively :

$$\underset{\sim}{S}(x) = \begin{cases} \dfrac{1}{b-a}(x-a) \text{ or } -\dfrac{1}{c-b}(x-b)+1, & x \in X^{+} \\ 0, & x \in X^{\circ} \\ \dfrac{1}{a}x - 1 \text{ or } -\dfrac{1}{d-c}(x-c), & x \in X^{-} \end{cases} \qquad (2.12)$$

$$\underset{\sim}{S}(x)=\begin{cases}\frac{1}{a}x & , x\in[0,a]\\ -\frac{1}{b-a}(x-a)+1 & , x\in(a,b]\\ -\frac{1}{c-b}(x-b) & , x\in(b,c]\\ \frac{1}{d-c}(x-c)-1 & , x\in(c,d]\end{cases} \tag{2.13}$$

B-B fuzzy kiss function and its graph in X given here play an important role in engineering fuzzy decision, engineering fuzzy control, especially linear B-B fuzzy kiss function. Due to limit of space and theme, these questions will be studied in later thesises which are omitted here.

3. Character Theorem of B-B Fuzzy Set $\underset{\sim}{S}$

Appointed that in the following : X is a finite universe, $\mathscr{F}$(X) is the set of B-B fuzzy set $\underset{\sim}{S}$ in X.

Definition 3.1 $\underset{\sim}{K}$ is said to be the subset of B-B fuzzy set $\underset{\sim}{S}$ in X and denoted by $\underset{\sim}{K}\subseteq\underset{\sim}{S}$, if fuzzy kiss measure of all $x\in$X concerning $\underset{\sim}{K}$, $\underset{\sim}{S}$ satisfies

$$\underset{\sim}{K}(x)\le\underset{\sim}{S}(x) \tag{3.1}$$

Due to different places of x in X, (3.1) can turn into one of (3.2), (3.3), (3.4):

1°. For any $x_i\in X^+\subset X$, $\underset{\sim}{K}^+\subseteq\underset{\sim}{S}^+$ then :

$$\underset{\sim}{K}^+(x_i)\ \le\ \underset{\sim}{S}^+(x_i) \tag{3.2}$$

2°. For any $x_j\in X^-\subset X$, $\underset{\sim}{K}^-\subseteq\underset{\sim}{S}^-$ then :

$$\underset{\sim}{K}^-(x_j)\ \le\ \underset{\sim}{S}^-(x_j) \tag{3.3}$$

Here, $\underset{\sim}{K}^+$, $\underset{\sim}{S}^+$ are B-B fuzzy sets of up-universe X^+, respectively ; $\underset{\sim}{K}^-$, $\underset{\sim}{S}^-$ are B-B fuzzy sets of down-universe X^-, respectively ; $\underset{\sim}{K}^\circ$, $\underset{\sim}{S}^\circ$ are B-B fuzzy sets of bound-universe X°, respectively.

3°. For any $x_k\in X^\circ\subset X$, $\underset{\sim}{K}^\circ\subseteq\underset{\sim}{S}^\circ$ then :

$$\underset{\sim}{K}^\circ(x_k)\ \le\ \underset{\sim}{S}^\circ(x_k) \tag{3.4}$$

Definition 3.2 $\underset{\sim}{S}^c$ is said to be the supplementary set of B-B fuzzy set $\underset{\sim}{S}$ in X, if fuzzy kiss measure of $x\in$X concerning $\underset{\sim}{S}^c$ satisfies

$$\underset{\sim}{S}^c(x)=\pm1\ -\ \underset{\sim}{S}(x) \tag{3.5}$$

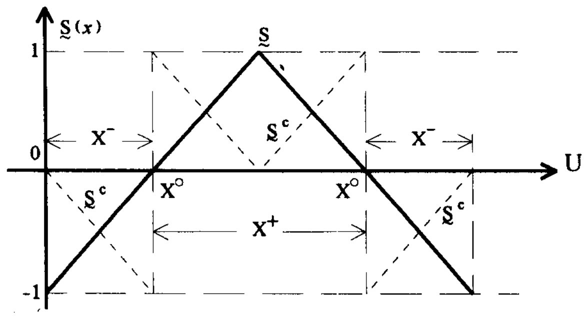
From (3.5) we get :

1°. if $x_i \in X^+$, (3.5) turns into

$$\underset{\sim}{S}^c(x_i) = 1 - \underset{\sim}{S}(x_i) \tag{3.6}$$

2°. if $x_j \in X^-$, (3.5) turns into

$$\underset{\sim}{S}^c(x_j) = -1 - \underset{\sim}{S}(x_j) \tag{3.7}$$



Graph 2.3 the graph of B-B fuzzy set $\underset{\sim}{S}$, $\underset{\sim}{S}^c$; $\underset{\sim}{S}$ is denoted by the solid line, $\underset{\sim}{S}^c$ is denoted by the imaginary line

Definition 3.3 $\underset{\sim}{S} \cup \underset{\sim}{Q}$ is said to be the union of B-B fuzzy set $\underset{\sim}{S}$, $\underset{\sim}{Q}$ in X, if fuzzy kiss measures of $x \in X$ concerning $\underset{\sim}{S}$, $\underset{\sim}{Q}$ satisfy

$$(\underset{\sim}{S} \cup \underset{\sim}{Q})(x) = \underset{\sim}{S}(x) \vee \underset{\sim}{Q}(x) \tag{3.8}$$

Due to different places in X of x, (3.8) can turn into one of (3.9), (3.10), (3.11):

1°. if $x_i \in X^+ \subset X$,

$$(\underset{\sim}{S}^+ \cup \underset{\sim}{Q}^+)(x_i) = \underset{\sim}{S}^+(x_i) \vee \underset{\sim}{Q}^+(x_i) \tag{3.9}$$

2°. if $x_j \in X^- \subset X$,

$$(\underset{\sim}{S}^- \cup \underset{\sim}{Q}^-)(x_j) = \underset{\sim}{S}^-(x_j) \vee \underset{\sim}{Q}^-(x_j) \tag{3.10}$$

3°. if $x_k \in X^\circ \subset X$,

$$(\underset{\sim}{S}^\circ \cup \underset{\sim}{Q}^\circ)(x_k) = \underset{\sim}{S}^\circ(x_k) \vee \underset{\sim}{Q}^\circ(x_k) \tag{3.11}$$

Definition 3.4 $\underset{\sim}{S} \cap \underset{\sim}{Q}$ is said to be the intersection of B-B fuzzy set $\underset{\sim}{S}$, $\underset{\sim}{Q}$ in X, if fuzzy kiss measures of $x \in X$ concerning $\underset{\sim}{S}$, $\underset{\sim}{Q}$ satisfy

$$(\underset{\sim}{S} \cap \underset{\sim}{Q})(x) = \underset{\sim}{S}(x) \wedge \underset{\sim}{Q}(x) \tag{3.12}$$

Definition 3.5 Let $\lambda \in [-1, 1]$, S_λ, $S_{\underset{\bullet}{\lambda}} \in \mathscr{P}(X)$ are said to be λ-cutset, λ-strong cutset of $\underset{\sim}{S} \in \mathscr{F}(X)$, respectively, where

$$S_\lambda = \{\, x \mid , \ |\underset{\sim}{S}(x)| \ge |\lambda| \,\} \tag{3.13}$$

$$S_{\dot{\lambda}} = \{\, x \mid , \ |\underset{\sim}{S}(x)| > |\lambda| \,\} \tag{3.14}$$

From (3.13), (3.14) we get λ-cutset S_λ^+, λ-strong cutset $S_{\dot{\lambda}}^+$ of $\underset{\sim}{S}$ in up-universe X^+ and λ-cutset S_λ^-, λ-strong cutset $S_{\dot{\lambda}}^-$ of $\underset{\sim}{S}$ in down-universe X^-, respectively.

1°. $S_\lambda^+ = \{x \mid , \ \underset{\sim}{S}^+(x) \ge \lambda \}$, $\quad S_{\dot{\lambda}}^+ = \{x \mid , \ \underset{\sim}{S}^+(x) > \lambda \}$

2°. $S_\lambda^- = \{x \mid , \ \underset{\sim}{S}^-(x) \le \lambda \}$, $\quad S_{\dot{\lambda}}^- = \{x \mid , \ \underset{\sim}{S}^-(x) < \lambda \}$

here, S_λ^+, $S_{\dot{\lambda}}^+ \subset X^+ \subset X$; S_λ^-, $S_{\dot{\lambda}}^- \subset X^- \subset X$.

From definition 3.1 ~ 3.5 we get

Theorem 3.1 Let $\underset{\sim}{S} \in \mathscr{F}(X)$, then

$$\underset{\sim}{S} \cup \underset{\sim}{S} = \underset{\sim}{S}, \quad \underset{\sim}{S} \cap \underset{\sim}{S} = \underset{\sim}{S}$$

Theorem 3.2 Let $\underset{\sim}{S}$, $\underset{\sim}{K} \in \mathscr{F}(X)$, then

$$\underset{\sim}{S} \cup \underset{\sim}{K} = \underset{\sim}{K} \cup \underset{\sim}{S}, \quad \underset{\sim}{S} \cap \underset{\sim}{K} = \underset{\sim}{K} \cap \underset{\sim}{S}$$

Theorem 3.3 Let $\underset{\sim}{S}$, $\underset{\sim}{K}$, $\underset{\sim}{Q} \in \mathscr{F}(X)$, then

$$(\underset{\sim}{S} \cup \underset{\sim}{K}) \cup \underset{\sim}{Q} = \underset{\sim}{S} \cup (\underset{\sim}{K} \cup \underset{\sim}{Q}), \quad (\underset{\sim}{S} \cap \underset{\sim}{K}) \cap \underset{\sim}{Q} = \underset{\sim}{S} \cap (\underset{\sim}{K} \cap \underset{\sim}{Q})$$

Theorem 3.4 Let $\underset{\sim}{S}$, $\underset{\sim}{K} \in \mathscr{F}(X)$, then

$$\underset{\sim}{S} \cap (\underset{\sim}{S} \cup \underset{\sim}{K}) = \underset{\sim}{S}, \quad \underset{\sim}{S} \cup (\underset{\sim}{S} \cap \underset{\sim}{K}) = \underset{\sim}{S}$$

Theorem 3.5 Let $\underset{\sim}{S}$, $\underset{\sim}{K}$, $\underset{\sim}{Q} \in \mathscr{F}(X)$, then

$$(\underset{\sim}{S} \cup \underset{\sim}{K}) \cap \underset{\sim}{Q} = (\underset{\sim}{S} \cap \underset{\sim}{Q}) \cup (\underset{\sim}{K} \cap \underset{\sim}{Q}), \quad (\underset{\sim}{S} \cap \underset{\sim}{K}) \cup \underset{\sim}{Q} = (\underset{\sim}{S} \cup \underset{\sim}{Q}) \cap (\underset{\sim}{K} \cup \underset{\sim}{Q})$$

Theorem 3.6 Let $\underset{\sim}{S} \in \mathscr{F}(X)$, then $\quad (\underset{\sim}{S}^c)^c = \underset{\sim}{S}$

Theorem 3.7 Let $\underset{\sim}{S}$, $\underset{\sim}{K} \in \mathscr{F}(X)$, then

$$(\underset{\sim}{S} \cup \underset{\sim}{K})^c = \underset{\sim}{S}^c \cap \underset{\sim}{K}^c, \quad (\underset{\sim}{S} \cap \underset{\sim}{K})^c = \underset{\sim}{S}^c \cup \underset{\sim}{K}^c$$

Theorem 3.8 Let $\underset{\sim}{S} \in \mathscr{F}(X)$, and appointed that $X^+(x) = 1$, $X^-(x) = -1$, $\varnothing(x) = 0$.

1. if $X = X^+ \cup X^\circ$, then

$$X \cap \underset{\sim}{S} = \underset{\sim}{S}, \quad X \cup \underset{\sim}{S} = X, \quad \varnothing \cap \underset{\sim}{S} = \varnothing, \quad \varnothing \cup \underset{\sim}{S} = \underset{\sim}{S}$$

2. if $X = X^- \cup X^\circ$, then

$$X \cap \underset{\sim}{S} = X, \quad X \cup \underset{\sim}{S} = \underset{\sim}{S}, \quad \varnothing \cap \underset{\sim}{S} = \underset{\sim}{S}, \quad \varnothing \cup \underset{\sim}{S} = \varnothing$$

Theorem 3.9 Let $\underset{\sim}{S}$, $\underset{\sim}{K}$, $\underset{\sim}{Q} \in \mathscr{F}(X)$, $\underset{\sim}{S} \subseteq \underset{\sim}{K}$, then

$$\underset{\sim}{S} \cup \underset{\sim}{Q} \subseteq \underset{\sim}{K} \cup \underset{\sim}{Q}, \quad \underset{\sim}{S} \cap \underset{\sim}{Q} \subseteq \underset{\sim}{K} \cap \underset{\sim}{Q}$$

Here we only give the proof of theorem 3.7, the proof of 3.1, 3.2, 3.3, 3.4, 3.5, 3.6, 3.8, 3.9 are omitted.

Proof of theorem 3.7 :

Proof : 1°. i) Let $x_i \in X^+ \cup X^\circ$, from definition 3.2, 3.3, $(\underset{\sim}{S} \cup \underset{\sim}{K})^c(x_i) = 1 - (\underset{\sim}{S} \cup \underset{\sim}{K})(x_i) = (1 - \underset{\sim}{S}(x_i)) \wedge (1 - \underset{\sim}{K}(x_i)) = \underset{\sim}{S}^c(x_i) \wedge \underset{\sim}{K}^c(x_i) \Rightarrow (\underset{\sim}{S} \cup \underset{\sim}{K})^c = \underset{\sim}{S}^c \cap \underset{\sim}{K}^c$. ii) Let $x_i \in X^- \cup X^\circ$, from definition 3.2, 3.3, $(\underset{\sim}{S} \cup \underset{\sim}{K})^c(x_i) = -1 - (\underset{\sim}{S} \cup \underset{\sim}{K})(x_i) = (-1 - \underset{\sim}{S}(x_i)) \wedge (-1 - \underset{\sim}{K}(x_i)) = \underset{\sim}{S}^c(x_i) \wedge \underset{\sim}{K}^c(x_i) \Rightarrow (\underset{\sim}{S} \cup \underset{\sim}{K})^c = \underset{\sim}{S}^c \cap \underset{\sim}{K}^c$. Due to i), ii) we get :

$$(\underset{\sim}{S} \cup \underset{\sim}{K})^c = \underset{\sim}{S}^c \cap \underset{\sim}{K}^c$$

2°. The proof which is omitted is similar to that of 1°.

In fact , we can get theorem 3.1 ~ 3.9 directly from B-B fuzzy kiss function given in graph 2.1.

Theorem 3.10 Let $\underset{\sim}{S}, \underset{\sim}{S}^{(t)} \in \mathscr{F}(X), t \in T$; then

$$\underset{\sim}{S} \cap (\bigcup_{t \in T} \underset{\sim}{S}^{(t)}) = \bigcup_{t \in T} (\underset{\sim}{S} \cap \underset{\sim}{S}^{(t)}), \quad \underset{\sim}{S} \cup (\bigcap_{t \in T} \underset{\sim}{S}^{(t)}) = \bigcap_{t \in T} (\underset{\sim}{S} \cup \underset{\sim}{S}^{(t)})$$

where $(\bigcap_{t \in T} \underset{\sim}{S}^{(t)})(x) = \bigwedge_{t \in T} \underset{\sim}{S}^{(t)}(x), \quad (\bigcup_{t \in T} \underset{\sim}{S}^{(t)})(x) = \bigvee_{t \in T} \underset{\sim}{S}^{(t)}(x)$

In fact , theorem 3.10 is the generalized form of theorem 3.5.

Theorem 3.11 Let $\underset{\sim}{S}, \underset{\sim}{K} \in \mathscr{F}(X), \lambda \in [-1, 1]$, then

$$(\underset{\sim}{S} \cup \underset{\sim}{K})_\lambda = S_\lambda \cup K_\lambda, \quad (\underset{\sim}{S} \cap \underset{\sim}{K})_\lambda = S_\lambda \cap K_\lambda, \quad (\underset{\sim}{S} \cup \underset{\sim}{K})_{\underset{\bullet}{\lambda}} = S_{\underset{\bullet}{\lambda}} \cup K_{\underset{\bullet}{\lambda}}, \quad (\underset{\sim}{S} \cap \underset{\sim}{K})_{\underset{\bullet}{\lambda}} = S_{\underset{\bullet}{\lambda}} \cap K_{\underset{\bullet}{\lambda}}$$

Theorem 3.12 Let $\underset{\sim}{S}^{(t)} \in \mathscr{F}(X), t \in T, \lambda \in [-1, 1]$; then

$$(\bigcup_{t \in T} \underset{\sim}{S}^{(t)})_\lambda \supseteq \bigcup_{t \in T} S_\lambda^{(t)}, \quad (\bigcap_{t \in T} \underset{\sim}{S}^{(t)})_\lambda = \bigcap_{t \in T} S_\lambda^{(t)}, \quad (\bigcup_{t \in T} \underset{\sim}{S}^{(t)})_{\underset{\bullet}{\lambda}} = \bigcup_{t \in T} S_{\underset{\bullet}{\lambda}}^{(t)}, \quad (\bigcap_{t \in T} \underset{\sim}{S}^{(t)})_{\underset{\bullet}{\lambda}} \subseteq \bigcap_{t \in T} S_{\underset{\bullet}{\lambda}}^{(t)}$$

Theorem 3.13 Let $\underset{\sim}{S} \in \mathscr{F}(X), \lambda \in [-1, 1]$; then $S_{\underset{\bullet}{\lambda}} \subseteq S_\lambda$

Theorem 3.14 Let $\underset{\sim}{S} \in \mathscr{F}(X), \lambda \in [-1, 1]$

1. if $\lambda_1 < \lambda_2$; $\lambda_1, \lambda_2 \in [0, 1]$, then $S_{\lambda_1} \supseteq S_{\lambda_2}$; $S_{\underset{\bullet}{\lambda_1}} \supseteq S_{\underset{\bullet}{\lambda_2}}$; $S_{\underset{\bullet}{\lambda_1}} \supseteq S_{\lambda_2}$
2. if $\lambda_1 < \lambda_2$; $\lambda_1, \lambda_2 \in [-1, 0]$, then $S_{\lambda_1} \subseteq S_{\lambda_2}$; $S_{\underset{\bullet}{\lambda_1}} \subseteq S_{\underset{\bullet}{\lambda_2}}$; $S_{\underset{\bullet}{\lambda_1}} \subseteq S_{\lambda_2}$

Theorem 3.15 Let $\underset{\sim}{S} \in \mathscr{F}(X), t \in T, \lambda_t \in [-1, 1]$, then $\bigcap_{t \in T} S_{\lambda_t} = S_{(\bigvee_{t \in T} \lambda_t)}$; $\bigcup_{t \in T} S_{\underset{\bullet}{\lambda_t}} = S_{(\underset{\bullet}{\bigwedge_{t \in T} \lambda_t})}$

Theorem 3.16 Let $\underset{\sim}{S}, \underset{\sim}{S}^c \in \mathscr{F}(X), \lambda \in [-1, 1]$; then $(\underset{\sim}{S}^c)_\lambda = (S_{\pm 1 - \underset{\bullet}{\lambda}})^c$; $(\underset{\sim}{S}^c)_{\underset{\bullet}{\lambda}} = (S_{\pm 1 - \lambda})^c$

Here we only give the proof of theorem 3.15, 3.16, the proof of 3.10, 3.11, 3.12, 3.13, 3.14 are omitted.

Proof of theorem 3.15 :

Proof : $1°$. i) Let $\lambda_t \in [0, 1]$, $t \in T$; $x \in \bigcap_{t\in T} S_{\lambda_t} \Leftrightarrow$ for any $t \in T$, $x \in S_{\lambda_t} \Leftrightarrow$ for any $t \in T$, $\underset{\sim}{S}(x) \geq \lambda_t$ $\Leftrightarrow \underset{\sim}{S}(x) \geq \bigvee_{t\in T} \lambda_t \Leftrightarrow x \in S_{(\bigvee_{t\in T} \lambda_t)}$. ii) Let $\lambda_t \in [-1, 0]$, $t \in T$, similar to i), we get $x \in \bigcap_{t\in T} S_{\lambda_t} \Leftrightarrow x \in S_{(\bigvee_{t\in T} \lambda_t)}$.

$2°$. The proof which is omitted is similar to that of $1°$.

Proof of theorem 3.16 :

Proof : $1°$. i) Let $\lambda \in [0, 1]$; $x \in (\underset{\sim}{S}^c)_\lambda \Leftrightarrow \underset{\sim}{S}^c(x) = 1 - \underset{\sim}{S}(x) \geq \lambda \Leftrightarrow \underset{\sim}{S}(x) \leq 1 - \lambda$ $\Leftrightarrow \underset{\sim}{S}(x) \not> 1 - \lambda \Leftrightarrow x \bar{\in} S_{1\dot{-}\lambda} \Leftrightarrow x \in (S_{1\dot{-}\lambda})^c$; ii) Let $\lambda \in [-1, 0]$, similar to i), we get $x \in (\underset{\sim}{S}^c)_\lambda \Leftrightarrow$ $x \in (S_{1\dot{-}\lambda})^c$.

$2°$. The proof which is omitted is similar to that of $1°$.

We point out that :

Theorem 3.1 ~ 3.16 given in this paper are obtained by B-B fuzzy set $\underset{\sim}{S}$ in universe $X = X^+ \cup X^- \cup X°$. If let $X^- = \varnothing$, then $X = X^+ \cup X^- \cup X°$ turns into $X = X^+ \cup X°$, B-B fuzzy set $\underset{\sim}{S}$ degenerates into O-B fuzzy set $\underset{\sim}{A}$, and all character theorems 3.1 ~ 3.16 concerning B-B fuzzy set $\underset{\sim}{S}$ in this paper are simplified into character theorems 3.1 ~ 3.16 of O-B fuzzy set $\underset{\sim}{A}$ which are known by people.

References

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