ON RESOLUTION PROCEDURES OF LATTICE-VALUED LOGICS BASED ON LATTICE IMPLICATION ALGEBRAS *

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Many-valued logic system always plays a crucial role in Artificial Intelligence. And automated theorem proving system has attached considerable attention because of its speed-up the development of Artificial Intelligence, especially the proving system based on non-classical logic. In this paper, we are concerned with the lattice-valued logic system with truth-value in lattice implication algebras which is for studying many-valued logic system and even the logic system based on lattice. Firstly, we analysis some limitations of classical resolution and some resolution procedures of fuzzy logic, then outline some prelinmary ideals of combining resolution procedure with the implication in lattice implication algebra and proposed a resolution-like procedure. Finally, some semantic issues are also discussed.

Many-valued logic system always plays a crucial role in AI. In order to study many-valued logic system and even the logic system based on lattice, an algebraic structure--lattice implication algebra was introduced in reference [1] and the corresponding lattice-valued logic system[2] were also established by the attempt of construct a formal theory to deal with uncertainty, vagueness information. In recent years, automated theorem proving system has attached considerable attention because it speed-up the development of AI, especially the proving system based on non-classical logic. In this paper, we are concerned with the lattice-valued logic system with truth-value in lattice implication algebras. Firstly, we analyzed some limitations of classical resolution and some resolution procedures of fuzzy logic, then outline some prelinmary ideals of combining resolution procedure with the implication in lattice implication algebra and a resolution-like procedure is proposed. Finally, some semantic issues are also discussed.

1 Preliminaries

Definition 1.1 Let $(L, \vee, \wedge, ')$ be a complemented lattice with the universal bounds $0, \mathrm{I}$, $\rightarrow$: $L \times L \rightarrow L$ be a mapping. $(L, \vee, \wedge, ', \rightarrow)$ is called a lattice implication algebra[1] if the following conditions hold for any $x, y, z \in L$:

$(\mathrm{I}_1)\ x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$

$(\mathrm{I}_2)\ x \rightarrow x = \mathrm{I}$

$(\mathrm{I}_4)\ x \rightarrow y = y \rightarrow x = \mathrm{I} \Rightarrow x \rightarrow y$

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(I_5) $(x \to y) \to y = (y \to x) \to x$

(k_1) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$

(k_2) $(x \wedge y) \to z = (x \to z) \vee (y \to z)$

Example 1.1[1] Let $(L, \vee, \wedge, ')$ be a Boolean lattice. If for any $x, y \in L$, $x \to y = x' \vee y$, then $(L, \vee, \wedge, ', \to)$ is a lattice implication algebra.,

Example 1.2[1] Let $L = [0, 1]$. If for any $x, y \in L$, $x \vee y = \max\{x, y\}$, $x \wedge y = \min\{x, y\}$, $x' = 1 - x$, $x \to y = 1 \wedge (1 - x + y)$, then $(L, \vee, \wedge, ', \to)$ is a lattice implication algebra.

Example 1.3 $\forall a, b$ $[0, 1]$, Jager[9] defined on the unit interval the following operations:

$$a' = (1 - a^p)^{1/p}, \qquad a \to b = 1 \wedge [1 - a^p + b^p]^{1/p}.$$

Then the structure $([0, 1], \vee, \wedge, \to, ', 0, 1)$ is for each fix $p \in \mathrm{N}$ a lattice implication algebra.

Example 1.4[9] Let f: $[0, 1] \to [0, k]$, $0 < k < \infty$, be a strictly increasing and continuous function with $f(0)=0$, $f(1)=k$. If we definie operation on [0, 1] by

$$a \to b = \begin{cases} f^{-1}(k - f(a) + f(b)) & \text{if } k - f(a) + f(b) \in [0, k]; \\ 1 & \text{otherwise.} \end{cases} \quad , \quad a = f(1) - f(a)).$$

Then we obtained a lattice implication algebra generated by the function.

From the above examples, we can notice that lattice implication algebra is a class of relatively general algebraic structure, at least including two important algebraic structures, i.e. Boolean algebra and Lukasiewicz algebra.

Since the celebrated resolution principle was introduced by J. P. Robinson in 1965, the problem is raised by an attempt of providing automated reasoning for the proposal of L. A. Zadeh of modeling vagueness through fuzzy sets. It was first studied by Lee and Chang[4, 5]. Afterwards, Mukaidono[6] proposed the concept of fuzzy resolution, Liu Xuhua introduced λ-resolution procedure etc.. In spite of the verifying and extending of the classical resolution, these proposal use only the so-called Kleene implication ($p \to q = p' \vee q$), which is the most straightforward generalization of classical implication. It implies that the formula of their logic are synatically equivalent to the formula in classical logic. Moreover, it is well known that different implications can capture various nuances of approximate reasoning can be brought out by considering as the degree of reliability --with which a certain formula q, Having a truth value $[q]$ has been inferred--the truth value of the conjunction of all the formulas used to infer it. Thus , for instance, if q is derived by modus ponens from p and $p \to q$, its degree of reliability is given by $[p \wedge (p \to q)]$, and q is the resolvent of the parent clause A and B, then its degree of reliability is given by $[A \wedge B]$. As is to expected,one can easily check the degree of reliability of formulas derivable in Kleene's system and in Lukasiewicz system do not, in general, coincide. Also in the light of semantical consideration, the development of a relatively efficient calculus for Lukasiewicz system seems desirable. In particular, the widespread use of resolution in automated theorem proving and its relatives (e.g. Prolog and Logic programming), and the central role that Lukasiewicz implication plays in all the developments and applications of fuzzy logic, suggest that the study of resolution procedure containing more general implication, especially the Lukasiewicz implication, can provide a useful framework for the development of approximate reasoning.

In essence, the resolution rule is a generalization of the Pule-literal rule and Modus Ponens rule, where $p \rightarrow q$ is defined as $p' \vee q$. So, implication operator is not involved in classical resolution procedure. However, it is just the implication operator that play an important role in lattice-valued logic based on lattice implication algebras and show an essential distinction with classical logic. Therefore, in order to establish resolution principle in lattice-valued logic system based on implication algebras, we should focus on finding a resolution procedure faced directly to implication operator in lattice implication algebras.

In what follows, we discuss some preliminary problems presented by the introduction of a variant of the classical resolution for a lattice-valued logic based on lattice implication algebras. For a standard presentation of resolution see, e.g., [3].

2 Preliminary ideals

Different from Kleene implication, the implication in lattice implication algebras can not transform into other classical connectives. It make the corresponding proof system more complex. In what follows, we shall outline some preliminary ideals for addressing these problems.

We introduced an operator "$\leftarrow$" in lattice-valued logic system LP(X) based on implication algebras as follows:

For any $p, q \in$ LP(X), $p \leftarrow q = (p \rightarrow q)'$.

It is easy to see that $p \rightarrow q = (p \leftarrow q)' = (p \leftarrow q)' \vee q$. The latter has a structure resembling $p \rightarrow q = p' \vee q$ by instead of the p' by the $(p \leftarrow q)'$. We could then try to paralleled the formalism of the resolution procedure by introducing a new type of formula $(p \leftarrow q)'$.

As the variant of classical resolution, a formula p in lattice-valued propositional logic system LP(X)[2] based on lattice implication algebras is semantically equivalent to p^* with only the connectives $'$, $\wedge$, $\vee$ and $\leftarrow$.

It is easy to see that "$\leftarrow$" satisfies the following properties:

1) $(p \leftarrow q)'' = p \leftarrow q$

2) $p \leftarrow (q \vee r) = (p \leftarrow q) \wedge (p \leftarrow r)$
3) $(p \vee q) \leftarrow r = (p \leftarrow r) \vee (q \leftarrow r)$
4) $p \leftarrow (q \wedge r) = (p \leftarrow q) \vee (p \leftarrow r)$
5) $(p \wedge q) \leftarrow r = (p \leftarrow r) \wedge (q \leftarrow r)$
6) $p \leftarrow q = q' \leftarrow p'$

By these properties together with the properties of $'$, $\vee$ and $\wedge$, arbitrary a formula p can be simplified into an extended conjunctive normal form, we defined as follows:

Definition 2.1 p and q are atoms, a formula as the form $p \leftarrow q$ is called a quasi-atom.

Definition 2.2 An atom or the negation of an atom is called literal. And the negation of quasi-atom is called quasi-literal.

Definition 2.3 The disjunction of literals and quasi-literals is called quasi-clause. The cinjunction of quasi-clauses is called quasi-conjunctive normal form.

From the properties of $\leftarrow$, $\wedge$, $\vee$ and $'$, a formula here can always be transformed into a quasi-conjunctive normal form

For a given quasi-conjunctive normal form, we introduce a possible resolution-like procedure similar to classical resolution on implication in lattice implication algebras, which allows classical resolution procedure, moreover,

1) MP: If F and G are two quasi-clauses such that F contains p and a formula in G contains $(p \leftarrow q)'$ as a subformula, then erase p from F and $(p \leftarrow q)'$ from G, writing q in its place. Finally, in accordance with the resolution procedure, takes the union of the resulting quasi-clause F′ and G′.

2) MT: If F contains q' and $(p \leftarrow q)'$ is a subformula of a formula in G, one is allowed to cancel them both and to write q in the place of $(p \leftarrow q)'$. Finally, in accordance with the resolution procedure, one takes the union of the resulting quasi-clause F′ and G′.

MP, when applicable, allows one to reduce the possible "$\leftarrow$" operator from quasi-clause to simplifies the proof procedure. In fact, when the implication operator can not be transformed into other classical connectives, MP provide a possible resolution-like procedure faced directly to implication for an attempt at simplifying the formulas with implication operator.

3 Semantic issues

It is well known that the management of truth values is a central issue in many-value logic, and the particular use which is made of them as a technical tool for modeling approximate reasoning. And it is the second aspect that induced us to consider the possibility of developing resolution-like procedures also in the case in which theimplication is different from classical implication. Indeed, different implications in a many-valued logic capture various facets of vagueness present in the description of states of affairs. Notice that the degree of reliability of a derivation of a given formula from certain premises is in general not the same if different implication are used. Although the proposed way of measuring the degree of reliability seems very natural, it remains an open problem to see whether other measures can be fruitfully used, and which form these measure can in general assume.

For example, in his study of the properties of Kleene's implication, Lee[5] show that the truth value of the resolvent is always greater than 0.5 in the case in which the truth values of the parent clauses are also greater than 0.5. This result is no more valid, as one can easily check, if Lukasiewicz's implication is considered. However, one may study the more general and quite open problem of which semantic relations hold between the premises and the conclusions of arbitrary a system of many-valued logic, and in the case of implication in lattice implication algebra, one may look for semantic properties similar to those found by Lee. Just to give an example as follows:

If $[(p \leftarrow q)' \vee s] \geqslant \alpha$, and $[p \vee r] \geqslant \beta$ then the clause obtained in our calculus by MP satisfies the following relation :

$$[q \vee r \vee s] \geqslant \alpha \otimes \beta \qquad (*)$$

where $[x]$, a number in the interval [0, 1], is the truth value assumed by x. α, $\beta \in L=[0, 1]$, $\alpha \otimes \beta = (\alpha \rightarrow \beta')'$.

We can check it by the following two results :

i) if $[s] \geqslant \alpha$ or $[r] \geqslant \beta$, then $[q \vee r \vee s] \geqslant \alpha \otimes \beta$ follows by the properties of $\otimes$ in lattice implication algebra.

ii) if the conditions of (i) are not hold, then we must have both $[(p \leftarrow q)'] \geqslant \alpha$ and $[p] \geqslant \beta$. Then we have $[q] \geqslant \alpha \otimes \beta$. In fact, by use of the properties of implication in reference [1], $([(p \leftarrow q)'] \otimes [p]) \rightarrow [q] = ([p \rightarrow q] \otimes [p]) \rightarrow [q] = [p \rightarrow q] \rightarrow ([p] \rightarrow [q]) = [p \rightarrow q] \rightarrow [p \rightarrow q] = 1$, i.e. $[q] \geqslant [(p \leftarrow q)'] \otimes [p] \geqslant \alpha \otimes \beta$.

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