SOME PROPERTIES OF FOUR VALUED PROPOSITIONAL LOGIC SYSTEM FL *

QIN KEYUN, XU YANG
Dept. of Applied Mathematics, Southwest Jiaotong University,
Chengdu 610031, Sichuan,, P.R.China

This paper is devoted to the study of a four valued propositional logic system FL This system was established by using the method of free algebra. In this system, the truth values of formulae can be incomparable. The semantical and syntactical problems of FL were discussed. The soundness theorem, deduction theorem were given.

In the study of lattice valued logic systems, the handle of incomparable truth values is of significance for the theory and application. But so far,there has no material progress for the study of incomparable truth values. In many famous finite valued logic systems, such as, Lukasiewicz system, Post system, sobocinski system and Godel system, the truth values domain are finite chain. In Pavelka's system, many important theorems were obtained by restriction that the truth values domain was a finite chain or unit interval [0,1]. In this paper, we establish a four valued propositional logic system, denoted by FL, and discuss some of its basic properties. In system FL, the truth values can be incomparable.

Let $L=\{O, a, b, I\}$. Defining binary operations $\vee$, $\wedge$, $\rightarrow$ and unary operation $'$ on L by the following:

$\vee$	0	a	b	I
0	0	a	b	I
a	a	a	I	I
b	b	I	b	I
I	I	I	I	I

$\wedge$	0	a	b	I
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
I	0	a	b	I

$\rightarrow$	0	a	b	I
0	I	I	I	I
a	b	I	b	I
b	a	a	I	I
I	0	a	b	I

x	0	a	b	I
x'	I	b	a	0

Then $(L, \vee, \wedge)$ is a lattice and $(L, \vee, \wedge, ', \rightarrow)$ is a lattice implication algebra[4]. O and I are the least and greatest elements of L respectively, a and b are incomparable.

Definition 1. Let X be the set of propositional variables, $T=L\cup\{\vee, \wedge, ', \rightarrow\}$ be a type with ar($\vee$)=ar($\wedge$)=ar($\rightarrow$)=2, ar($'$)=1 and ar(x)=0 for each $x\in L$, the propositional

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algebra of four valued propositiona logic system FL is the free T algebra on X and denoted by FL(X).

Definition 2. A valuation of FL(X) is a T algebra homomorphism γ from FL(X) to L.

If γ is a valuation of FL(X), then $\gamma(x)=x$ for each $x\in L$.

Since X is the set of propositional variables, the values $\gamma(x)$ for $x\in X$ may be assigned arbitrarily. These values, once assigned, determine the homomorphism γ uniquely and so determine $\gamma(p)$ for all $p\in$ FL(X).

Denote the set of all L-type fuzzy sets on FL(X) by F_L(FL(X)).

Definition 3. Let $A\in F_L$(FL(X)) and γ be a valuation of FL(X), γ is said to satisfy A if for each $p\in$ FL(X), $A(p)\leq\gamma(p)$. A is said to be satisfiable if there exist valuation γ such that γ satisfy A.

Definition 4. Let $A\in F_L$(FL(X)), $p\in$ FL(X), $\alpha\in L$. We say that A semantically implies p by truth value α if $\gamma(p)\geq\alpha$ for each valuation γ which satisfies A. We shall write this as $A\models_\alpha p$.

If $\varnothing\models_\alpha p$, we say that p is valid by truth value α and write this as $\models_\alpha p$. If $\models_I p$, that is to say that $\gamma(p)$=I for each valuation γ of FL(X), then p is called valid and denoted by $\models p$.

Theorem 1. For each $p, q, r\in$ FL(X), the following formulaes are valid:

(1) $p\to I$,

(2) $p\to p$,

(3) $(p\to q)\to((q\to r)\to(p\to r))$,

(4) $(p\to(q\to r))\to(q\to(p\to r))$,

(5) $p\wedge q\to p$,

(6) $q\wedge p\to p$,

(7) $(p\to r)\to((p\to q)\to(p\to r\wedge q))$,

(8) $p\to p\vee q$,

(9) $p\to q\vee p$,

(10) $(p\to r)\to((q\to r)\to(p\vee q\to r))$,

(11) $p\to(q\to p\otimes q)$,

(12) $p\to(q\to p\wedge q)$,

(13) $p\to(q\to p)$,

(14) $(p\to(q\to r))\to((p\to q)\to(p\to r))$,

(15) $(p\to(q\to r))\to(p\otimes q\to r)$,

(16) $(p\to q)\to(q'\to p')$,

(17) $(p')'\to p$

where $p\otimes q=(p\to q')'$.

Let $A_L \in F_L(FL(X))$ satisfy the following condition: for each $p \in FL(X)$,

$$A_L(\varphi)=\begin{cases} 1, & \varphi \textit{ is the formula as the form } (1), \cdots (17) \\ x, & \varphi \textit{ is } x,\ x \in L \\ 0, & \textit{otherwise} \end{cases}$$

Then A_L is called the axiom of system FL.

Definition 5. Let $A \in F_L(FL(X))$, $\phi \in FL(X)$. A formal proof ω of ϕ from A is a finite sequence of the following form:

$$\omega\!: (\phi_1, \alpha_1), (\phi_2, \alpha_2), \cdots, (\phi_n, \alpha_n)$$

where $\phi_n=\phi$. For each i, $1 \le i \le n$, $\phi_i \in FL(X)$, $\alpha_i \in L$ and

(1) $A_L(\phi_i)=\alpha_i$, or

(2) $A(\phi_i)=\alpha_i$, or

(3) there exist j, $k < i$ such that $\phi_j=\phi_k \to \phi_i$ and $\alpha_i=\alpha_j \otimes \alpha_k$, or

(4) there exist $j < i$ and $\alpha \in L$ such that $\phi_i=\alpha \to \phi_j$ and $\alpha_i=\alpha \to \alpha_j$.

where $\alpha_j \otimes \alpha_k=(\alpha_j \to \alpha_k')'$.

In this definition, n is called the length of ω and denoted by $l(\omega)$, α_n is called the value of ω and denoted by $\mathrm{val}(\omega)$.

Definition 6. Let $A \in F_L(FL(X))$, $\phi \in FL(X)$, $\alpha \in L$, ϕ is called an α theorem of A, denoted by $A \mid\!-_{\alpha} \phi$, if

$$\alpha \le \vee\{\mathrm{val}(\omega);\ \omega \text{ is a formal proof of } \phi \text{ from } A\}.$$

If $\varnothing \mid\!-_{\alpha} \phi$, we say that ϕ is an α theorem and write this simply as $\mid\!-_{\alpha} \phi$. If $\mid\!-_{1} \phi$, then ϕ is called a theorem and write simply as $\mid\!- \phi$.

Let $\mathrm{Con}: F_L(FL(X)) \to F_L(FL(X))$ be defined as

$$\mathrm{Con}(A)(\phi)=\vee\{\mathrm{val}(\omega);\ \omega \text{ is a formal proof of } \phi \text{ from } A\}$$

for each $A \in F_L(FL(X))$ and $\phi \in FL(X)$. It can be easily proved that Con is a closure operator.

Theorem 2. Let $A \in F_L(FL(X))$, $\phi \in FL(X)$, $\alpha \in L$, γ is a valuation of $FL(X)$, if $A \mid\!-_{\alpha} \phi$ and γ satisfy A, then $\gamma(\phi) \ge \alpha$.

Proof. For each formal proof ω of ϕ from A:

$$\omega\!: (\phi_1, \alpha_1), (\phi_2, \alpha_2), \cdots, (\phi_n, \alpha_n)$$

we shall prove by inductive method that $\gamma(\phi) \ge \mathrm{val}(\omega)$

1. If $l(\omega)=1$, then $A(\phi_n)=\alpha_n$ or $A_L(\phi_n)=\alpha_n$

(1) if $A(\phi_n)=\alpha_n$, then

$$\gamma(\phi)=\gamma(\phi_n) \ge A(\phi_n)=\alpha_n=\mathrm{val}(\omega).$$

(2) if $A_L(\phi_n)=\alpha_n$, then by $A_L \subseteq \gamma$ we have

$$\gamma(\phi)=\gamma(\phi_n) \ge A_L(\phi_n)=\alpha_n=\mathrm{val}(\omega).$$

2. Suppose that the conclusion holds for each formal proof of ϕ from A whose

length is less than n, then for ω

(1) if $A(\phi_n)=\alpha_n$ or $A_L(\phi_n)=\alpha_n$, then the conclusion can be proved similarly as that of 1.

(2) if there exist i,j< n such that $\phi_i=\phi_j\to\phi_n$ and $\alpha_n=\alpha_i\otimes\alpha_j$, by inductive hypothesis, $\gamma(\phi_i)\geq\alpha_i$, $\gamma(\phi_j)\geq\alpha_j$, and hence

$$\gamma(\phi)=\gamma(\phi_n)\geq\gamma(\phi_j\to\phi_n)\otimes\gamma(\phi_j)$$
$$=\gamma(\phi_i)\otimes\gamma(\phi_j)\geq\alpha_i\otimes\alpha_j=\alpha_n=\mathrm{val}(\omega)$$

(3) if there exist i< n and $\alpha_0\in L$ such that $\phi_n=\alpha_0\to\phi_i$ and $\alpha_n=\alpha_0\to\alpha_i$, by inductive hypothesis, $\gamma(\phi_i)\geq\alpha_i$ and hence

$$\gamma(\phi)=\gamma(\phi_n)=\gamma(\alpha_0\to\phi_i)=\gamma(\alpha_0)\to\gamma(\phi_i)$$
$$=\alpha_0\to\gamma(\phi_i)\geq\alpha_0\to\alpha_i=\alpha_n=\mathrm{val}(\omega)$$

So we have proved $\gamma(\phi)\geq\mathrm{val}(\omega)$ and hence

$$\gamma(\phi)\geq\vee\{\mathrm{val}(\omega);\ \omega \text{ is a formal proof of } \phi \text{ from } A\}\geq\alpha.$$

Corollary 1(Soundness theorem). Let $A\in F_L(FL(X))$, $\phi\in FL(X)$, $\alpha\in L$, if $A|-_\alpha\phi$, then $A|=_\alpha\phi$.

Theorem 3(Deduction theorem). Let A, $A^*\in F_L(FL(X))$, ϕ, $\psi\in FL(X)$, $\alpha\in L$, $A^*=A\cup\{1/\phi\}$, then $A|-_\alpha\phi\to\psi$ if and only if $A^*|-_\alpha\psi$.

Proof.

Necessity: By $A|-_\alpha\phi\to\psi$ and $A\subseteq A^*$, we have

$$\alpha\leq\vee\{\mathrm{val}(\omega);\ \omega \text{ is a formal proof of } \phi\to\psi \text{ from } A\}$$
$$=\mathrm{Con}(A)(\phi\to\psi)\leq\mathrm{Con}(A^*)(\phi\to\psi)$$

that is to say $A^*|-_\alpha\phi\to\psi$ and hence $A^*|-_\alpha\psi$ by $A^*|-_1\phi$.

Adequacy: (1) We first prove that for each formal proof ω_ψ of ψ from A^*, there exist a formal proof $\omega_{\phi\to\psi}$ of $\phi\to\psi$ from A such that $\mathrm{val}(\omega_{\phi\to\psi})\geq\mathrm{val}(\omega_\psi)$.

In fact,for formal proof ω_ψ,

① if $l(\omega_\psi)=1$, let ω_ψ be (ψ,a), then $A^*(\psi)=a$ or $A_L(\psi)=a$.

If $A_L(\psi)=a$, then the sequence

(ψ, a)

$(\psi\to(\phi\to\psi), 1)$

$(\phi\to\psi, a)$

is a formal proof of $\phi\to\psi$ from A, denote it by $\omega_{\phi\to\psi}$, then

$$\mathrm{val}(\omega_{\phi\to\psi})=\mathrm{val}(\omega_\psi)$$

If $A^*(\psi)=a$, then the conclusion can be proved similarly.

② Suppose that the conclusion holds for every formal proof of ψ from A^* whose length is less than m, then for the formal proof ω_ψ

$$(\psi_1, a_1), (\psi_2, a_2), \cdots, (\psi_m, a_m)$$

where $\psi_m=\psi$ and $a_m=\mathrm{val}(\omega_\psi)$

a) If $A^*(\psi_m)=a_m$ or $A_L(\psi_m)=a_m$, then the conclusion can be proved similarly as that of ①.

b) If there exist i, j<m such that $\psi_i=\psi_j\rightarrow\psi_m$ and $a_m=a_i\otimes a_j$, by inductive hypothesis, there exist formal proof of $\phi\rightarrow\psi_i$ from A:

$(p_1, b_1), (p_2, b_2), \cdots, (p_s, b_s)$

such that $a_i \leq b_s$; there exist formal proof of $\phi\rightarrow\psi_j$ from A:

$(q_1, c_1), (q_2, c_2), \cdots (q_t, c_t)$

such that $a_j\leq c_t$, so the sequence

$(p_1, b_1), (p_2, b_2), \cdots, (p_s, b_s), (q_1, c_1), (q_2, c_2), \cdots, (q_t, c_t),$

$((\phi\rightarrow(\psi_j\rightarrow\psi_m))\rightarrow((\phi\rightarrow\psi_j)\rightarrow(\phi\rightarrow\psi_m)), I)$

$((\phi\rightarrow\psi_j)\rightarrow(\phi\rightarrow\psi_m), b_s)$

$(\phi\rightarrow\psi_m, b_s\otimes c_t)$

is a formal proof of $\phi\rightarrow\psi$ from A and

$\text{val}(\omega_\psi)=a_m=a_i\otimes a_j\leq b_s\otimes c_t$

c) If there exist i<m, $a_0\in L$ such that $\psi_m=a_0\rightarrow\psi_i$ and $a_m=a_0\rightarrow a_i$, then by inductive hypothesis, there exist formal proof of $\phi\rightarrow\psi_i$ from A:

$(p_1, b_1), (p_2, b_2), \cdots, (p_s, b_s)$

such that $a_i\leq b_s$, so the sequence

$(p_1, b_1), (p_2, b_2), \cdots (p_s, b_s),$

$(a_0\rightarrow(\phi\rightarrow\psi_i), a_0\rightarrow b_s),$

$((a_0\rightarrow(\phi\rightarrow\psi_i))\rightarrow(\phi\rightarrow(a_0\rightarrow\psi_i)), I),$

$(\phi\rightarrow(a_0\rightarrow\psi_i), a_0\rightarrow b_s)$

is a formal proof of $\phi\rightarrow\psi_m$ from A and

$\text{val}(\omega_\psi)=a_m=a_0\rightarrow a_i\leq a_0\rightarrow b_s$

(2) If $A^*|\!-_\alpha \psi$, by (1)

$\alpha\leq\vee\{\text{val}(\omega);\ \omega \text{ is a formal proof of } \psi \text{ from } A^*\}$

$\leq\vee\{\text{val}(\omega);\ \omega \text{ is a formal proof of } \phi\rightarrow\psi \text{ from } A\}$

that is to say, $A|\!-_\alpha \phi\rightarrow\psi$.

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