ON THE PRIME SPACE OF LIA*

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In this paper we introduced the concept of prime space of lattice implication algebra, in which we analysised its topological property and discussed the relation between the category of topological space and the category of lattice implication algebras.

1. Introduction

People have paid more attention to lattice logic system, which will become much logical system for interlligent computer. In reference [1] Xu Yang present an algebra structure — lattice implication algebra by combing the lattice with implication algebra as truth values domain of lattice valued logical system. After that, we study the implication homomorphism, congruence relations and algebraic structure of lattice implication algebra and discuss the first order logical system FM based on lattice implication algebras, and obtained several importance results. The implication filter, which derived from modus ponens in logic, play an important role in lattice implication algebra. The purpose of this paper is as follows: After discussing the prime implication filter of lattice implication algebra, we introduced the concept of prime space of lattice implication algebra, in which we analysised its topological properties and discussed the relation between the category of topological space and the category of lattice implication algebras.

2. Preliminary concepts

Definition 2.1 Let $(L, \vee, \wedge, ')$ be a complemented lattice with the universal bounds 0, I, $\rightarrow$: $L \times L \rightarrow L$ be a mapping. $(L, \vee, \wedge, ', \rightarrow)$ is called a lattice implication algebra[1] (shortly as LIA) if the following conditions hold for any $x, y, z \in L$:

(I_1) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$

(I_2) $x \rightarrow x = I$

(I_3) $x \rightarrow y = y' \rightarrow x'$

(I_4) $x \rightarrow y = y \rightarrow x = I \Rightarrow x \rightarrow y$

(I_5) $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$

(k_1) $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$

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(k_2) $(x\wedge y)\to z=(x\to z)\vee(y\to z)$

Definition 2.2 Let $(L, \vee, \wedge, ', \to)$ be a lattice implication algebra. $J\subseteq L$ is said to be a implication filter of L, if it satisfies the following conditions:

(1) $I\in L$.

(2) For all, $x, y\in L$, if x and $x\to y\in L$, then $y\in L$.

Properties of LIA as concerned can be seen in reference [1-2].

Definition 2.3 Let $(L, \wedge, \vee, ', \to)$ be a lattice implication algebra, F be a proper implication filter of L. If for arbitrary a, $b\in L$, $a\vee b\in F$ implies $a\in F$ or $b\in F$. Then F is called as prime implication filter of L.

Let $F(L)$ denote all implication filters of lattice implication algebra L, and $\mathbf{PF}(L)$ denote all prime implication filters of L. We have known that $(F(L), \wedge, \vee)$ is a complete distributive lattice, where for arbitrary A, $B\in F(L)$, $A\wedge B=A\cap B$, $A\vee B=[A\cup B)$ (the smallest implication filter including $A\cup B$).

Theorem 2.1 Let $F\in F(L)-\{L\}$. Then $F\in\mathbf{PF}(L)$ if and only if for arbitrary a, $b\in L$, $a\to b\in F$ or $b\to a\in F$.

Proof. Let $F\in\mathbf{PF}(L)$. For arbitrary a, $b\in L$, since $(a\to b)\vee(b\to a)=I\in F$, so $a\to b\in F$ or $b\to a\in F$.

Conversely, for arbitrary a, $b\in L$, assume that $a\vee b\in F$. If $a\to b\in F$, then from $(a\to b)\to b=a\vee b\in F$, and note that F is a implication filter, we have $b\in F$; Similarly, $b\to a\in F$ implies $a\in F$, hence $F\in\mathbf{PF}(L)$.

Theorem 2.2 Let $F\in\mathbf{PF}(L)$. Then for arbitrary $a\in L$, at most one of a and a' belongs to F.

Remark If $a\in L$ is a Boolean element, i. e. $a\vee a'=I$, and if $F\in\mathbf{PF}(L)$, then the only one of a and a' belongs to F.

Theorem 2.3 Let $F\in F(L)$, $a\in L$ but $a\notin F$. Then there exists $P_a\in\mathbf{PF}(L)$, such that $a\notin P_a$ and $F\subseteq P_a$.

Proof. Let $A=\{P;\ P\in F(L)$ and $a\notin P,\ F\subseteq P\}$. Since $F\in A$, so $A\neq\varnothing$. By using of Zorn lemma, it is easy to prove that : There exists maximal element in A in the including order "$\subseteq$" . Let P_a be a maximal element of A. Then $a\notin P_a\in F(L)$. In the following, wwill prove that P_a is a prime implication filter. It is enough to prove that for arbitrary $x, y\in L$, if $x\vee y\in P_a$, then $x\in P_a$ or $y\in P_a$.

In fact, because $x\vee y\in P_a$, we can obtain that

$$[P_a\cup\{x\})\cap[P_a\cup\{y\})=P_a.$$

From $a\notin P_a$, we know that: $a\notin[P_a\cup\{x\})$ or $a\notin[P_a\cup\{y\})$. Moreover since P_a is a maximal element, we have: $[P_a\cup\{x\})=P_a$ or $[P_a\cup\{y\})=P_a$, i.e., $x\in P_a$ or $y\in P_a$.

Corollary 2.1 For arbitrary $a\in L$ and $a\neq I$, there exists $P\in\mathbf{PF}(L)$, such that $a\notin P_a$.

3. Prime space of lattice implication algebras

Let L be a lattice implication algebra. For arbitrary $a\in L$, Denote $X_a=\{F|F\in \mathbf{PF}(L), a\notin F\}$. And let $X=\{X_a \mid a\in L\}$.

From Corollary 2.1, we know that: If $a\neq I$, $X_a\neq\phi$. And obviously $X_I=\phi$, $X_0=\mathbf{PF}(L)$.

Lemma 3.1 $\bigcup\limits_{a\in L} X_a = \mathbf{PF}(L)$.

Proof. For arbitrary $F\in\mathbf{PF}(L)$, since F is a proper implication filter, we know that there exists $a\in L$ such that $a\notin F$, so $F\in X_a \subseteq \bigcup\limits_{a\in L} X_a$. It following that $\mathbf{PF}(L)\subseteq \bigcup\limits_{a\in L} X_a$. And obviously $\bigcup\limits_{a\in L} X_a = \mathbf{PF}(L)$. Hence $\bigcup\limits_{a\in L} X_a = \mathbf{PF}(L)$.

From Lemma 3.1, $X=\{X_a \mid a\in L\}$ is a covering of $\mathbf{PF}(L)$. Then by taking X as a subbase we can introduce a topology $\Im$ on $\mathbf{PF}(L)$, such that ($\mathbf{PF}(L)$, $\Im$) become a topological space. We call it as prime space of L. In the following, we will discuss the structure and properties of this topological space.

For arbitrary $F\in F(L)$, Let $\overline{F} = \{P|P\in\mathbf{PF}(L), F\not\subset P\}$.

Lemma 3.2 1. For arbitrary $a\in L$, $\overline{[a)} = X_a$.

2. For arbitrary $M, N\in F(L)$, $\overline{M}\cap\overline{N} = \overline{M\cap N}$.

3. For arbitrary $G_\lambda\in F(L)$, $\lambda\in\Gamma$, $\overline{\bigvee\limits_{\lambda\in\Gamma} G_\lambda} = \bigcup\limits_{\lambda\in\Gamma}\overline{G_\lambda}$.

Proof. 1. For arbitrary $P\in\mathbf{PF}(L)$, since $[a)\not\subset P$ if and only if $a\notin P$, so $\overline{[a)} = X_a$.

2. For arbitrary $P\in\mathbf{PF}(L)$, let $M\not\subset P$, $N\not\subset P$. Then there exist $m\in M$, $n\in N$ such that $m, n\notin P$, and notice that P is a prime implication filter, M and N is implication filter, we have: $m\vee n\in M\cap N$. Hence $M\cap N\not\subset P$, $P\in\overline{M\cap N}$, that is, $\overline{M}\cap\overline{N}\subseteq\overline{M\cap N}$. The opposite inclusion is obvious. Consequently, $\overline{M}\cap\overline{N} = \overline{M\cap N}$.

3. For arbitrary $P\in\mathbf{PF}(L)$, if for every $\lambda\in\Gamma$, $G_\lambda\subseteq P$, then $\bigvee\limits_{\lambda\in\Gamma} G_\lambda\subseteq P$, that is, if $\bigvee\limits_{\lambda\in\Gamma} G_\lambda\not\subset P$, there exists $\lambda_0\in\Gamma$ such that $G_\lambda\not\subset P$. So $\overline{\bigvee\limits_{\lambda\in\Gamma} G_\lambda}\subseteq\bigcup\limits_{\lambda\in\Gamma}\overline{G_\lambda}$. And the opposite inclusion is obvious. Consequently, $\overline{\bigvee\limits_{\lambda\in\Gamma} G_\lambda} = \bigcup\limits_{\lambda\in\Gamma}\overline{G_\lambda}$.

Lemma 3.3 Let $F\in F(L)$. Then $a\in F$ if and only if $X_a\subseteq\overline{F}$.

Proof. If $a\in F$, then for arbitrary $P\in X_a$, $a\notin P$. It follows that $P\not\subset F$, $P\in\overline{F}$. So $X_a\subseteq\overline{F}$. If $a\notin F$, from Theorem 2.3, there exists $P_a\in\mathbf{PF}(L)$ such that $a\notin P_a$ and $F\subseteq P_a$, so $P_a\in X_a$ but $P_a\notin\overline{F}$. Hence $X_a \subseteq\overline{F}$, that is, if $X_a\subseteq\overline{F}$, it must have $a\in F$.

Theorem 3.1 For arbitrary $F\in F(L)$, $\overline{F}$ is an open set of prime space ($\mathbf{PF}(L)$, $\Im$), and every open set of prime space can be uniquely expressed as the form of $\overline{F}$.

Proof. At first, for arbitrary $F \in F(L)$, $F = \vee_{a \in F} [a)$. And it follows from Lemma 3.2 that: $\overline{F} = \overline{\vee_{a\in F}[a)} = \bigcup_{a\in F}\overline{[a)} = \bigcup_{a\in F} X_a \in \Im$.

Moreover, from $[a) \cap [b) = [a \vee b)$ and Lemma 3.2, it is easy to see that $\overline{F}$ is just the arbitrarily union of finitely inter section of the subsets of $\{X_a \mid a \in F\}$. It means that $\{\{\overline{P} \mid P \in F(L)\}$ are just right open set families of prime space, that is $\Im = \{\overline{P} \mid P \in F(L)\}$.

Then we prove the expression of open set is unique. In fact, from Lemma 3.3, for arbitrary $F \in F(L)$, $a \in F$ if and only if $X_a \subseteq \overline{F}$. So if $\overline{F} = \overline{P}$, then we have $a \in F$ if and only if $X_a \subseteq \overline{F} = \overline{P}$ if and only if $a \in P$, that is, $F=P$.

Theorem 3.2 X_a is the only compact-open set of prime space.

Proof. From the definition, X_a is a open set. To prove compact, in view of Theorem 3.1 we may assume that $\{\overline{G} \mid G_\lambda \in F(L),\ \lambda \in \Gamma\}$ is a arbitrary open covering of X_a, i.e. $X_a \subseteq \bigcup_{\lambda\in\Gamma}\overline{G_\lambda} = \overline{\vee_{\lambda\in\Gamma} G_\lambda}$.

From Lemma 3.3, we know $a \in \vee_{\lambda\in\Gamma} G_\lambda$. Moreover, according to the structure of generated implication filter, we know that there exists finite subset $\Gamma_0 \subseteq \Gamma$ such that $a \in \vee_{\lambda\in\Gamma} G_\lambda$. By Lemma 3.3, $X_a \subseteq \overline{\vee_{\lambda\in\Gamma} G_\lambda} = \bigcup_{\lambda\in\Gamma}\overline{G_\lambda}$. This means that X_a is a compact set.

Now, we assume that $\overline{F}$ ($F \in F(L)$) is arbitrary a compact-open set. Since $\overline{F} = \bigcup_{a\in F} X_a$, so $\{X_a \mid a \in F\}$ is a open-covering of $\overline{F}$. It follows that there exist a_1, a_2, ……, $a_n \in F$ such that

$$\overline{F} = \bigcup_{i=1}^{n} X_{a_i} = \bigcup_{i=1}^{n}\overline{[a)} = \overline{\vee_{i=1}^{n}[a_i)} = \overline{[\wedge_{i=1}^{n} a_i)} = X_{\wedge_{i=1}^{n} a_i}$$

Consequently, X_a is the only one compact-open set.

Theorem 3.3 $X=\{X_a \mid a \in L\}$ is the open-set base of prime space.

Because of $X_0=\mathbf{PF}(L)$ and that X_0 is a compact set, we have:

Theorem 3.4 Prime space is a compact space.

Theorem 3.5 For arbitrary $P \in \mathbf{PF}(L)$, there must have $cl(\{P\}) = \mathbf{PF}(L) - \overline{P}$ in prime space, where $cl(\{P\})$ expresses the closure of single element set $\{P\}$.

Proof. Let note that $Q \in cl(\{P\})$ if and only if every neighborbood of Q contains P. And $X=\{X_a \mid a \in L\}$ is a base, so the neighborbood can be limited to X_a, and, hence

$$\begin{aligned}\mathrm{cl}(\{P\}) &= \{Q|Q\in \mathbf{PF}(L) \text{ and for every } a\in L,\ Q\in X_a \text{ implies } P\in X_a\}\\ &= \{Q|Q\in \mathbf{PF}(L) \text{ and for every } a\in L,\ a\notin Q \text{ implies } a\notin P\}\\ &= \{Q|Q\in \mathbf{PF}(L) \text{ and } P\subseteq Q\}\\ &= \mathbf{PF}(L)-\overline{P}\end{aligned}$$

Theorem 3.6 Prime space is a T_0 -space.

4. The category of lattice implication algebras and that of topological spaces

We denote lattice implication algebra as LIA[1], topological space as TOP[4]. For arbitrary $A\in$LIA, let T_A denote the prime space of A, that is, T_A=**PF**(A).

Lemma 4.1 Let A, $B\in$LIA, $F\in$Hom(A, B). If $F\in$**PF**(B), then $f^{-1}(B)=\{x|x\in A$ and $f(x)\in B\}\in$**PF**(A).

In view of Lemma 4.1, we may definite the following mappings T between the category LIA and the category TOP:

The correspond between the objects: $T(A)=\mathcal{T}_A$, $A\in$LIA.

The correspond between the arrows-sets: for arbitrary A, $B\in$LIA, for arbitrary $f\in$Hom(A, B), $T(f)$: $\mathcal{T}_B\to\mathcal{T}_A$,

$$f\to f^{-1}(F),$$

Lemma 4.2 Let A, $B\in$LIA, $f\in$Hom(A, B). Then $T(f)\in$Hom($\mathcal{T}_A$, $\mathcal{T}_B$).

Proof. To prove $F(f)\in$Hom(O_B, O_A), that is, $T(f)$ is a countinous mapping from T_B to T_A.. It is enough to prove that for arbitrary $a\in A$, $(T(f))^{-1}(X_a)\in T$, where T express the family of open sets of prime space. In fact, $P\in X_{f(a)}$ if and only if $f(a)\notin P$ if and only if $a\notin f^{-1}(P)$ if and only if $f^{-1}(P)\in X_a$ if and only if $P\in(F(f))^{-1}(X_a)$. It follows that $(F(f))^{-1}(X_a)=X_{f(a)}\in T_B$. .

It is easy to prove that $T(f)\circ T(g)=T(F\circ g)$ and $F(I_A)=I_{F(A)}$. Hence:

Theorem 4.1 T is a contravariant functor from LIA to TOP, and F is a covariant functor from LIA^{OP} to TOP.

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