The chain structure of fuzzy subgroups

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Abstract: The subgroup and normal subgroup of a fuzzy group are defined and their basic properties are studied.The structure of the normal series of a fuzzy group is also discussed.
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1.Introduction

The concept of fuzzy group was formulated by Rosenfeld in [5] and has been investigated by several researchers who not only defined new concepts but also extended many results of classical group theorey to the fuzzy setting. In this paper we study the relations among the fuzzy groups by define the subgroup and normal subgroup of a fuzzy group.We also research the normal series and composition series of fuzzy groups and generalize some classical results to fuzzy group theory.The most important tool we use in this paper is the level subgroup of a fuzzy group.

2.Preliminaries

In this paper G will always denote a classsical group.

Definition 2.1 Let A be a fuzzy subset on G.If for any $x,y \in G$,

1) $A(xy) \geqslant \min\{A(x),A(y)\}$;
2) $A(x^{-1}) \geqslant A(x)$;

we call A the fuzzy subgroup of G.

If $A(xy) = A(yx)$,then A is called the fuzzy noemal subgroup of G.

Proposition 2.2 A fuzzy subset A of G is a fuzzy subgroup of G iff for any $\alpha \in [0,1]$,if A_α is nonempty,then A_α is a subgroup of G where A_α is called the level subgroup of A.

Definition 2.3 Let A,B be fuzzy subsets of nonempty set G, then the product of A and B is defined as $AB(x) = \sup_{x=ab} \min\{A(a),B(b)\}, x \in G.$

Definition 2.4 Let A be a fuzzy set of G.If for any nonempty set $T \subseteq G$,there exists $x_0 \in T, A(x_0)=\sup_{x \in T} A(x)$,then we call A has the sup property.

Proposition 2.5 Let A,B be fuzzy subsets of G. If A, B have the sup property,then $(AB)_\alpha = A_\alpha B_\alpha$ holds for any $\alpha \in [0,1]$.

3.The subgroup and normal subgroup of a group

Let A,B be fuzzy subgroups of G.In generally,if $B(x) \leqslant A(x)$,for any $x \in G$,we call B the subgroup of A,Although B is the subgroup of A,it always is not the substructure of A and it is possible that the structure of B is more complex than the structure of A. So we try to redefine the subgroup of a fuzzy group to avoid the above aspects.

Definition 3.1 Let A,B be fuzzy subgroups of G.If for any $\alpha \in [0,1]$, $B_\alpha \subseteq A_\alpha$,and if $B_\alpha \neq B_\beta$,then $A_\alpha \neq A_\beta$,for any $\alpha, \beta \in [0,1]$, then we call B the subgroup of A.If there existes a(for any)$\alpha \in [0, 1]$ satisfying $\{e\} \subset B_\alpha \subset A_\alpha$,then we call B the non- trivial(real) subgroup of A.

Proposition 3.2 If B is the subgroup of A,then:

(1) $B(x) \leqslant A(x)$,for any $x \in G$;

(2) there is a one to one mapping from Im(B) onto a subset of Im(A).

Proposition 3.3 Let B be a subgroup of A.If A has the sup property, then B has it.

Proposition 3.4 A classical group G has a fuzzy subgroup A which has non-trivial subgroup iff G is not a group with prime order.
The proof is clear.

Proposition 3.5 If B is a real subgroup of A,then there existes $x_0 \neq e$, and $B(X_0) < A(x_0)$.

Definition 3.6 Let B be a subgroup of A,and for any $\alpha \in [0,1]$,$B_\alpha \neq \phi$, B_α is the normal subgroup of A_α ,then we call B is the normal subgroup of A.

Proposition 3.7 If B is the subgroup of A and B is the fuzzy normal subgroup of G, then B is the normal subgroup of A.
Proof If B is the fuzzy normal subgroup of G, then for any $\alpha \in [0,1]$, $B_\alpha \neq$, B_α is the mormal subgroup of G, hence B_α is the normal subgroup of A_α, hence B is the normal subgroup of A.

It is obviously that as the same as in the classical group theory, if B is the normal subgroup of A, A is the subgroup of A′, then B may not be the normal subgroup of A′.

Proposition 3.8 Let B be the subgroup of A. Then B is the normal subgroup of A iff for any $x,y \in G, \alpha \in [0,1]$, if $A(x) \geqslant \alpha$, $B(y) \geqslant \alpha$, then $B(x^{-1}yx) \geqslant \alpha$.
Proof If $A(x) \geqslant \alpha$, $B(y) \geqslant \alpha$, then $x \in A_\alpha$, $y \in B_\alpha$. Since B is the normal subgroup of A, B_α is the normal subgroup of A_α, hence $x^{-1}yx \in B_\alpha$, that is to say $B(x^{-1}yx) \geqslant \alpha$.

If for any $x,y \in G, \alpha \in [0,1], A(x) \geqslant \alpha, B(y) \geqslant \alpha$ follows $B(x^{-1}yx) \geqslant \alpha$, it is clear B_α is the normal subgroup of A_α, hence B is the normal subgroup of A.

Proposition 3.9 If B,C are normal subgroup of A,A has the sup property, then both BC and $B \cap C$ are normal subgroup of A.
Proof Since both $B_\alpha C_\alpha$ and $B_\alpha \cap C_\alpha$ is the normal subgroups of A_α, so the proof is clear

4.The chain structure of fuzzy subgroup

In this section we let A(e)=B(e) for any fuzzy subgroups A,B of G.
Let A be the fuzzy subgroup of G, E_A is the fuzzy subgroup of G so that $E_A(e) = A(e)$ and $E_A(x) = 0$ for any other $x \in G$. It is clear that E_A is the smallest subgroup of A.

Definition 4.1 Let $A = A^{(1)} \supseteq A^{(2)} \supseteq \ldots \supseteq A^{(s+1)} = E_A$ be a subgroup series of A and A^{i+1} is the normal subgroup of A^i, then we call it the fuzzy normal series of A.

Definition 4.2 Let A has two fuzzy normal series as following:

$$A = A^{(1)} \supseteq A^{(2)} \supseteq \ldots \supseteq A^{(s+1)} = E_A, \qquad (1)$$

$$A = H^{(1)} \supseteq H^{(2)} \supseteq \cdots \supseteq H^{(t+1)} = E_A, \qquad (2)$$

If for any $\alpha \in [0,1]$, $A_\alpha = A_\alpha^{(1)} \supseteq A_\alpha^{(2)} \supseteq \ldots \supseteq A_\alpha^{(s+1)} = \{e\}$ and $A_\alpha = H_\alpha^{(1)} \supseteq H_\alpha^{(2)} \supseteq \ldots \supseteq H_\alpha^{(t+1)} = \{e\}$ are equivalent, then we call (1) and (2) are equivalent .We say that one fuzzy normal series is a refinement of a second if its terms include all of the fuzzy groups that occur in tne second series.

In the classical group throry,the equivalence of two normal series is expressed by the factor group.But in the fuzzy group theory we can not deal it with the fuzzy factor group since the fuzzy factor group has not been discussed so well to be a tool.

Proposition 4.3 Two fuzzy normal series are equivalent iff for any $\alpha \in [0,1]$,it is possible to set up a 1-1 correspondence between the factors of $A_\alpha = A_\alpha^{(1)} \supseteq A_\alpha^{(2)} \supseteq \dots \supseteq A_\alpha^{(s+1)} = \{e\}$ and $A_\alpha = H_\alpha^{(1)} \supseteq H_\alpha^{(2)} \supseteq \dots \supseteq H_\alpha^{(t+1)} = \{e\}$ such that the paired factors are isomorphic.

Proof By the definition of the equivalence of two classical normal series the proof is clear.

Proposition 4.4 If A has the sup property, then two fuzzy normal series of A have equivalent refinements

Proof Let $A = A^{(1)} \supseteq A^{(2)} \supseteq \dots \supseteq A^{(s+1)} = E_A$ and $A = H^{(1)} \supseteq H^{(2)} \supseteq \dots \supseteq H^{(t+1)} = E_A$ be two fuzzy normal series of A.

Let $A^{(ik)} = (A^{(i)} \cap H^{(k)})A^{(i+1)}$, $k = 1,2,\dots,t+1$,

$H^{(ki)} = (A^{(i)} \cap H^{(k)})H^{(k+1)}$, $i = 1,2,\dots,s+1$.

Since A has the sup property,then

$$[(A^{(i)} \cap H^{(k)})A^{(i+1)}]_\alpha = (A_\alpha^{(i)} \cap H_\alpha^{(k)})A_\alpha^{(i+1)},$$
$$[(A^{(i)} \cap H^{(k)})H^{(k+1)}]_\alpha = (A_\alpha^{(i)} \cap H_\alpha^{(k)})H_\alpha^{(k+1)}.$$

We have

$$A = A^{(11)} \supseteq A^{(12)} \supseteq \dots \supseteq A^{(1,t+1)}$$
$$= A^{(21)} \supseteq A^{(22)} \supseteq \dots \supseteq A^{(2,t+1)} \supseteq \dots \supseteq A^{(s,t+1)} = E_A;$$
$$A = H^{(11)} \supseteq H^{(12)} \supseteq \dots \supseteq H^{(1,s+1)}$$
$$= H^{(21)} \supseteq H^{(22)} \supseteq \dots \supseteq H^{(2,s+1)} \supseteq \dots \supseteq H^{(t,s+1)} = E_A.$$

By the Schreier's refinement theorem in the classical group theory we know the above two fuzzy normal series are equivalent.

Proposition 4.5 If $A = A^{(1)} \supseteq A^{(2)} \supseteq \dots \supseteq A^{(s+1)} = E_A$ is a fuzzy normal series of A,H is a subgroup of A,then

$$H = H \cap A^{(1)} \supseteq H \cap A^{(2)} \supseteq \dots \supseteq H \cap A^{(s+1)} = E_H$$

is a fuzzy normal series of H.

The proof is clear.

Definition 4.6 Let B be a subgroup of A and $B \neq E_A$,$B \neq A$.If there has not a non-trivial normal subgroup S of A satisfying $B \subset S \subset A$,then we call B the maximal normal subgroup of A.

Proposition 4.7 B is a maximal normal subgroup of A iff for any $\alpha \in [0,1]$, $A_\alpha \neq \phi$, if $B_\alpha \neq A_\alpha$, then A_α / B_α is simple.

Proof Sufficiency If B is a maximal normal subgroup of A, then we can not find a non-trivial normal subgroup S of A such that $B \subset S \subset A$ holds. That is to say for any $\alpha \in [0,1]$, there has not a normal subgroup H of A_α such that $B_\alpha \subset H \subset A_\alpha$ which implies $B_\alpha = A_\alpha$ or B_α is the maximsal normal subgroup of A_α which indicates A_α / B_α is simple.

Necessity is clear.

Lemma 4.8 B is a real maximal normal subgroup of A iff for any $\alpha \in [0,1]$, $A_\alpha \neq \phi$, A_α / B_α is simple.

The proof is clear.

Definition 4.9 Let $A = A_1 \supset A_2 \supset \dots \supset A_{n+1} = E_A$ be a normal serires of A satisfying A_{i+1} is the real maximal normal subgroup of A_i, then we call it a composotion series for A.

Proposition 4.10 Let A has the sup property, then any two composition series for a fuzzy group A are equivalent.

Proof By the definitions of the maximal normal subgroup and the composotion series we know $A = A^{(1)} \supset A^{(2)} \supset \dots \supset A^{(n+1)} = E_A$ is a composition series for A iff for any $\alpha \in [0,1]$, if $A_\alpha \neq \phi$, then $A_\alpha = A_\alpha^{(1)} \supset A_\alpha^{(2)} \supset \dots \supset A_\alpha^{(n+1)} = \{e\}$ is a composition series of A_α.

Hence by the Jordan – Holder Theorem in the classical group theory we know for any $\alpha \in [0,1]$, if $A_\alpha \neq \phi$, then any two composition series of A_α are equivalent. By the definition of the equivalence of fuzzy normal series we know any two composition series for A are equivalent.

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