Fuzzy Weakly Urysohn Spaces

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Abstract

In this paper, we introduce the concept of fuzzy weakly Urysohn space and establish some of its properties in fuzzy topological spaces.

Key words: Fuzzy topological spaces, Fuzzy weakly Urysohn spaces, Fuzzy strongly semiopen sets, Romote-neighborhoods.

1 Introduction

In 1968, Chang first established the concept of fuzzy topological spaces [4] based on Zadeh's concept of fuzzy sets [10]. Later Pu and Liu introduced the concept of the Q-neighborhood [6] and Wang introduced the concept of the remote-neighborhood [8] . Since this fuzzy topology has developed considerably. Chen generalized the Urysohn space [3] to the fuzzy topological spaces in [5].In this paper we introduce the concept of fuzzy weakly Urysohn space with the help of the remote-neighborhood [8] and the fuzzy strong semiinterior of fuzzy sets [1]. It is observed that

fuzzy Urgsohn space
$\Rightarrow$ fuzzy weakly Urysohn space
$\Rightarrow$ fuzzy Hausdorff space [8].

2 Preliminaries

In this paper, (X, δ) will denote a fuzzy topological space. A^0, A^- and A' will denote respectively the interior, closure and complement of the fuzzy set A.

Definition 2.1 [8]. Let (X, δ) be a fuzzy topological space, e be a fuzzy point, $P \in \delta'$ and $e \not\leq P$. Then P is called a remote-neighborhood of e, and the set of all remote-neighborhood of e will be denoted by $\eta(e)$.

Definition 2.2 [8]. A fuzzy topological space (X, δ) is called fuzzy Hausdorff if for every pair of fuzzy points x_α and y_λ with $x \neq y$ there are $P \in \eta(x_\alpha)$ and $Q \in \eta(y_\lambda)$ such that $P \cup Q = 1$.

Definition 2.3 [5]. A fuzzy topological space (X, δ) is called fuzzy Urysohn if for every pair of fuzzy points x_α and y_λ with $x \neq y$ there are $P \in \eta(x_\alpha)$ and $Q \in \eta(y_\lambda)$ such that $P^0 \cup Q^0 = 1$.

Definition 2.4 [1]. Let A be a fuzzy set of a fuzzy topological space (X, δ). A is called fuzzy strongly semiopen iff there is a $B \in \delta$ such that $B \leq A \leq B^{-0}$. A is called fuzzy strongly semiclosed iff there is a $B \in \delta'$ such that $B^{0-} \leq A \leq B$. $A^\triangle = \cup\{B|B \leq A,\ B$ fuzzy strongly semiopen$\}$ and $A^\sim = \cap\{B|B \geq A, B$ fuzzy strongly semiclosed$\}$ are called the strongly semiinterior and the strongly semiclosure of A, respectively.

3 Fuzzy weakly Urysohn spaces

Definition 3.1. A fuzzy topological space (X, δ) is called a fuzzy weakly Urysohn space if for every pair of fuzzy points x_α and y_λ with $x \neq y$ there are $P \in \eta(x_\alpha)$ and $Q \in \eta(y_\lambda)$ such that $P^\triangle \cup Q^\triangle = 1$.

Obviously, every fuzzy Urysohn space is a fuzzy weakly Urysohn space and every fuzzy weakly Urysohn sapce is a fuzzy Hausdorff space.

Theorem 3.2. *A fuzzy topoliogical space* $(X,\ \delta)$ *is a fuzzy weakly Urysohn space iff for every pair of fuzzy points* x_α *and* y_λ *with* $x \neq y$ *and* $\alpha, \lambda \in [0, 1)$, *there exist* $V \in \delta, W \in \delta$ *so that* $x_\alpha \in V, y_\lambda \in W$ *and* $V^\sim \cap W^\sim = 0$.

Proof. Let $(X,\ \delta)$ be a fuzzy weakly Uryshon space, x_α and y_λ be two fuzzy points in X with $x \neq y$ and $\alpha, \lambda \in [0, 1)$. Choose two real numbers s and t satisfying $0 < s < 1-\alpha$ and $0 < t < 1-\lambda$. Then there are romote-neighborhoods $P \in \eta(x_s)$ and $Q \in \eta(y_t)$ such that $P^\triangle \cup Q^\triangle = 1$. Put $V = P'$ and $W = Q'$. Then $V \in \delta, W \in \delta$, and $x_\alpha \in V, y_\lambda \in W$ and

$$V^\sim \cap W^\sim = P'^\sim \cup Q'^\sim = P^{\triangle\prime} \cap Q^{\triangle\prime} = (P^\triangle \cup Q^\triangle)' = 0.$$

Conversely, let the given condition hold. Suppose x_α and y_λ are two fuzzy points with $x \neq y$. Choose two real numbers s and t satisfying $1 - \alpha < s < 1$ and $1 - \lambda < t < 1$. In the light of the assumption, there exist $V \in \delta, W \in \delta$ so that $x_s \in V, y_t \in W$ and $V^\sim \cap W^\sim = 0$. Put $P = V'$ and $Q = W'$. Then $P \in \eta(x_\alpha), Q \in \eta(y_\lambda)$ and

$$P^\triangle \cup Q^\triangle = V'^\triangle \cup W'^\triangle = V^{\sim\prime} \cup W^{\sim\prime} = (V^\sim \cap W^\sim)' = 1.$$

Hence $(X,\ \delta)$ is a fuzzy weakly Urysohn space.

Definition 3.3. Let $(X,\ \delta)$ be a fuzzy topological sapce, e be a fuzzy point and $S = \{S(n), n \in D\}$ a fuzzy net [8] in X. Then e is said to be a w-limit point of S (or S w-converges to e) if S is eventually not in $P^\triangle$ for each $P \in \eta(e)$.

Theorem 3.4. *A fuzzy topological space* $(X,\ \delta)$ *is a fuzzy weakly Urysohn space iff no fuzzy net in* X *can w-converge to two fuzzy points* x_α *and* y_λ *with* $x \neq y$.

Proof. Let $(X,\ \delta)$ be a fuzzy weakly Urysohn space, $S = \{S(n), n \in D\}$ be a fuzzy net in X which w-converges to a fuzzy point x_α, and y_λ be another fuzzy point with $x \neq y$. Then there are $P \in \eta(x_\alpha)$ and $Q \in \eta(y_\lambda)$ such that $P^\triangle \cup Q^\triangle = 1$. Since S is eventually not in $P^\triangle$, therefore, S is eventually in $Q^\triangle$.

Hence S does not w-converge to y_λ.
Conversely, assume that the condition is true and that x_α and y_λ are two fuzzy points with $x \neq y$. If for every $P \in \eta(x_\alpha)$ and $Q \in \eta(y_\lambda), P^\triangle \cup Q^\triangle \neq 1$, then there exists a fuzy point $S(P,Q) \notin P^\triangle \cup Q^\triangle$. Take

$$S = \{S(P,Q) : (P,Q) \in \eta(x_\alpha) \times \eta(y_\lambda)\},$$

then S is a net in X with the following relation:

$$(P_1,\ Q_1) \leq (P_2,\ Q_2) \text{ iff } P_1 \subset P_2 \text{ and } Q_1 \subset Q_2$$

where $(P_1,\ Q_1), (P_2,\ Q_2) \in \eta(x_\alpha) \times \eta(y_\lambda)$. Obviously, S is eventually not in $P^\triangle$, so S w-converges to x_α. Similarly, S w-converges to y_λ as well. This contradicts the hypothesis. Consequently, there are $P \in \eta(x_\alpha)$ and $Q \in \eta(y_\lambda)$ such that $P^\triangle \cup Q^\triangle = 1$. Thus $(X,\ \delta)$ is a fuzzy weakly Urysohn space.

Definition 3.5. A fuzzy topological space $(X,\ \delta)$ is called strongly semi-interior additive if $(A \cup B)^\triangle = A^\triangle \cup B^\triangle$ for any two fuzzy sets A and B in X.

Theorem 3.6. *If $(X,\ \delta)$ is a Hausdorff and strongly semiinterior additive fuzzy topological space, then $(X,\ \delta)$ is a fuzzy weakly Urysohn space.*

Proof. Let x_α and y_λ be two fuzzy points in $(X,\ \delta)$ with $x \neq y$. According to Definition 2.2, there are $P \in \eta(x_\alpha)$ and $Q \in \eta(y_\lambda)$ such that $P \cup Q = 1$. Since $(X,\ \delta)$ is strongly semiinterior additive,

$$P^\triangle \cup Q^\triangle = (P \cup B)^\triangle = 1.$$

Hence $(X,\ \delta)$ is a fuzzy weakly Urysohn space.

Lemma 3.7. *Let $f : (x,\ \delta) \mapsto (Y,\ \tau)$ be a fuzzy homeomorphic mapping [9, 10] from a fuzzy space $(X,\ \delta)$ to another fuzzy space $(Y,\ \tau)$. If A is a fuzzy strongly semiopen set of $(X,\ \delta)$, then $f(A)$ is a strongly semiopen set of $(Y,\ \tau)$.*

Proof. Let A be a fuzzy strongly semiopen set of $(X,\ \delta)$. Then there is a $B \in \delta$ such that $B \leq A \leq B^{-0}$. Hence

$$f(B) \leq f(A) \leq f(B^{-0}).$$

Since f is a fuzzy homeomorphic mapping, $f(B) \in \tau$ and

$$f(B^{-0}) = (f(B))^{-0},$$

i.e., there is a $f(B) \in \tau$ such that

$$f(B) \leq f(A) \leq (f(B))^{-0}.$$

Thus $f(A)$ is a fuzzy strongly semiopen set of $(Y,\ \tau)$.

The following theorem shows that the fuzzy weakly Urysohn's separation is fuzzy homeomorphic invariance.

Theorem 3.8. *Let $f : (x,\ \delta) \mapsto (Y,\ \tau)$ be a fuzzy homeomorphic mapping from a fuzzy weakly Urysohn space $(X,\ \delta)$ to another fuzzy space $(Y,\ \tau)$. Then $(Y,\ \tau)$ is also a fuzzy weakly Urysohn space.*

Proof. Let y_α and y^*_λ be two fuzzy points in $(Y,\ \tau)$ with $y \neq y^*$. Then there are two fuzzy points x_α and x^*_λ in $(X,\ \delta)$ with $x \neq x^*$ such that $f(x_\alpha) = y_\alpha$ and $f(x^*_\lambda) = y^*_\lambda$. Since $(X,\ \delta)$ is a fuzzy weakly Urysohn space, there are $P \in \eta(x_\alpha)$ and $Q \in \eta(x^*_\lambda)$ such that $P^\triangle \cup Q^\triangle = 1_X$. Because f is fuzzy homeomorphic, we have $f(P) \in \eta(y_\alpha)$ and $f(Q) \in \eta(y^*_\lambda)$. By lemma 3.7

$$\begin{aligned}(f(P))^\triangle \cup (f(Q))^\triangle &\geq \left(f(P^\triangle)\right)^\triangle \cup \left(f(Q^\triangle)\right)^\triangle \\ &= f(P^\triangle) \cup f(Q^\triangle) \\ &= f(P^\triangle \cup Q^\triangle) \\ &= f(1_X) \\ &= 1_Y.\end{aligned}$$

Thus $(Y,\ \tau)$ is a fuzzy weakly Urysohn space.

References

[1] Bai Shi-Zhong, Fuzzy strongly semiopen sets and fuzzy strong semicontinuity, Fuzzy Sets and Systems 52(1992) 345-351.

[2] Bai Shi-Zhong, Q-convergence of nets and weak separation axioms in fuzzy lattices, Fuzzy Sets and Systems 88(1997)379-386.

[3] M.P.Berri, J.R.Porter and R.M.Stephenson, Jr., A survey of minimal topological spaces, Greneral Topology and its Relations to Modern Analysis and Algebra. III, Academia, Prague, 1971, 93-114.

[4] C.L.Chang, Fuzzy topological spaces, J.Math. Anal. Appl. 24(1968) 182-190.

[5] Chen Shui-Li, Fuzzy Urysohn spaces and α-stratified fuzzy Urysohn spaces, Fifth IFSA world congress (1993) 453-456.

[6] Pu Bao-Ming and Liu Ying-Ming, Fuzzy topology I, J.Math. Anal. Appl. 76(1980)571-599.

[7] Pu Bao-Ming and Liu Ying-Ming, Fuzzy topology II, J.Math. Anal. Appl. 77(1980), 20-37.

[8] Wang Guo-Jun, A new fuzzy compactness defined by fuzzy nets, J.Math.Anal.Appl. 94(1983)1-23.

[9] Wang Guo-Jun, Theory of L-fuzzy topological spaces, Pross of Shaanxi Normal University, Xian, 1988.

[10] L.A.Zadeh, Fuzzy sets, Inform. and Control 8(1965)338-353.