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HELLY'S THEOREM IN BANACH LATTICES WITH ORDER CONTINUOUS NORMS.

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ABSTRACT. We present Helly and Helly-Bray theorems in the context of vector lattices with order continuous norm. The corresponding generalized moment theorem and a corollary on representation of positive operators are stated and proved.

1. Introduction

Helly's theorem had been of some importance along time above all in the probability theory in connection with a problem of moments of distributions. As for other applications, we shall return to them later.

Recall that in this connection, real-valued nondecreasing functions f on the interval $[\alpha, \beta]$ of the real line are considered and that the following facts are true (see [**7**, Chapter 3]).

(1) The function f has at most countably many points of discontinuity on $[\alpha, \beta]$.
(2) (*Helly's theorem*) Given a uniformly bounded sequence (f_n) of real-valued nondecreasing functions on $[\alpha, \beta]$, there exists a subsequence (f_{n_k}) of (f_n) converging in some sense to a real-valued nondecreasing function f.
(3) (*Helly-Bray theorem*) Given a sequence (f_n) of real-valued nondecreasing functions on $[\alpha, \beta]$, converging in some sense to a real-valued nondecreasing function f, then for every continuous real-valued function g on $[\alpha, \beta]$, we have

$$\lim_{n\to\infty} \int_a^b g(t)\, df_n(t) = \int_a^b g(t)\, df(t).$$

We shall investigate these three properties for functions f and f_n with values in Banach lattices. We shall see that they do not remain true for any Banach lattice and that we must confine ourselves to the narrower class of Banach lattices with order continuous norm. Next we shall give two applications of these investigations : a generalized moment theorem and a representation theorem.

2. Helly and Helly-Bray theorems

We recall that a Banach lattice E is said to have an *order continuous norm* if $\lim_\alpha \|x_\alpha\| = 0$, for every nonincreasing net (x_α) in E such that $0 = \inf x_\alpha$.

As a part of [**1**, Theorem 12.9], we have the following result.

Proposition 1. *The Banach lattice E has order continuous norm if and only if each order interval in E is weakly compact. Moreover, a Banach lattice with order continuous norm is necessarily Dedekind complete.*

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The equivalence of (i) and (ii) in the next proposition is the main result of [**8**]. Note that we shall only use the implication (i) $\Rightarrow$ (ii) (that is a direct application of [**9**, III.2, Theorem 3]).

Proposition 2. *Given a σ-Dedekind complete Banach lattice E, the following conditions are equivalent :*

(i) E has order continuous norm ;

(ii) Every nondecreasing function $f\ : [0,1] \to E$ has at most countably many points of discontinuity.

Proposition 3. *For a nondecreasing function f defined on an interval I of $\mathbb{R}$, with values in a Banach lattice E with order continuous norm, and for $x \in I$, the following conditions are equivalent :*

(i) $f(x) = \inf\{f(y) \mid x < y \in I\}$;

(ii) f is right-continuous at x for the norm topology ;

(iii) f is right-continuous at x for the weak topology.

We have a similar characterization for the left-continuity and, in particular, for continuity.

Proof. (i) $\Rightarrow$ (ii) is a direct consequence of the order continuity of the norm, (ii) $\Rightarrow$ (iii) is obvious and (iii) $\Rightarrow$ (i) is due to the fact that the positive cone of E is closed for the weak topology. $\square$

Let us now state and prove an Helly's theorem in the setting of Banach lattices.

Theorem 1 (Helly's theorem). *Let $[\alpha,\beta]$ be a closed interval in $\mathbb{R}$. For a Banach lattice E, the following conditions are equivalent :*

(1) E has order continuous norm ;

(2) If $(f_n)_{n\in\mathbb{N}}$ is a sequence of nondecreasing functions on $[\alpha,\beta]$, with values in some order interval $[a,b]$ in E, then there exists a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ of (f_n) and a nondecreasing function $f\ : [\alpha,\beta] \to [a,b]$ such that $(f_{n_k}(x))_{k\in\mathbb{N}}$ is convergent to $f(x)$ for the weak topology $\sigma(E,E')$ at each continuity point $x \in]\alpha,\beta[$ of f, but also for $x=\alpha$ and for $x=\beta$.

Moreover, if E has order continuous norm, then the function f in (2) can be assumed to be right-continuous at every point $x \in]\alpha,\beta[$.

Proof. (1) $\Rightarrow$ (2). Let $(\alpha_k)_{k\in\mathbb{N}}$ be a dense sequence in $[\alpha,\beta]$ including α and β. Since the order interval $[a,b]$ is weakly compact (Proposition 1), the sequence (f_n) has a subsequence $(f_n^{(1)})$ such that $(f_n^{(1)}(\alpha_1))_{n\in\mathbb{N}}$ is weakly convergent to some $a_1 \in [a,b]$ (by Eberlein-Smulian theorem [**1**, Th. 10.13]). By induction, for $p = 2,3,\dots$, the sequence $(f_n^{(p-1)})$ has a subsequence $(f_n^{(p)})$ such that $(f_n^{(p)}(\alpha_p))_{n\in\mathbb{N}}$ is weakly convergent to some $a_p \in [a,b]$. Using the well known diagonal process, we define a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ of (f_n) by $f_{n_k} = f_k^{(k)}$. It is clear that the sequence $(f_{n_k}(\alpha_p))_{k\in\mathbb{N}}$ is weakly convergent to a_p $(p = 1,2,\dots)$ and, the considered functions being nondecreasing, that $\alpha_r \leq \alpha_s$ implies $a_r \leq a_s$.

Recalling that E is Dedekind complete (Proposition 1), we now define a nondecreasing function $f\ : [\alpha,\beta] \to [a,b]$ by

$$f(x) = \inf\{a_r \mid x \leq \alpha_r\}.$$

It is clear that $\lim_{k\to\infty} f_{n_k}(x) = f(x)$ for $\sigma(E,E')$ if $x=\alpha$ or $x=\beta$. Let us show that this equality remains true for any $x \in]\alpha,\beta[$ such that f is continuous at x. To this end, we recall first that the topological dual E' of E is the set of all differences of two positive linear functionals on E [1, Corollary 12.5]. Let φ be any positive linear functional on E. For any $\alpha_p \le x$, we have $\varphi(f_{n_k}(\alpha_p)) \le \varphi(f_{n_k}(x))$ and, since $(f_{n_k}(\alpha_p))_{k\in\mathbb{N}}$ is weakly convergent to $f(\alpha_p) = a_p$, we obtain :

$$\varphi(f(\alpha_p)) \le \liminf_{k\to\infty} \varphi(f_{n_k}(x)).$$

But $\{\alpha_p \mid \alpha_p \le x\}$ is a directed upwards set converging to x and, by continuity of f at x, $\varphi(f(x))$ is the limit of the net $\{\varphi(f(\alpha_p)) \mid \alpha_p \le x\}$. It follows that

$$\varphi(f(x)) \le \liminf_{k\to\infty} \varphi(f_{n_k}(x))$$

and, similarly, by considering $\alpha_p \ge x$, we also obtain

$$\limsup_{k\to\infty} \varphi(f_{n_k}(x)) \le \varphi(f(x)).$$

We conclude that $\varphi(f(x)) = \lim_{k\to\infty}\varphi(f_{n_k}(x))$ and, finally, that $f(x)$ is the weak limit of the sequence $(f_{n_k}(x))_{k\in\mathbb{N}}$.

To prove that we may assume f right-continuous at every point $x \in]\alpha,\beta[$, it suffices to replace f by the function $g : [\alpha,\beta] \to [a,b]$, defined by

$$\begin{cases} g(\alpha) = f(\alpha),\ g(\beta) = f(\beta); \\ g(x) = \inf\{f(y) \mid x < y \in [\alpha,\beta]\} \quad \text{for } x \in]\alpha,\beta[. \end{cases}$$

It is obvious that $f \le g$, g is nondecreasing, right-continuous on $]\alpha,\beta[$ and also that $f(x) = g(x)$ for some $x \in]\alpha,\beta[$ if and only if f is right-continuous at x. We now show that f is continuous at $x \in]\alpha,\beta[$ if and only if g is continuous at x. If f is continuous at x, we have successively

$$g(x) = f(x) = \sup\{f(y) \mid y < x\} \le \sup\{g(y) \mid y < x\}$$

and, consequently,

$$g(x) = \sup\{g(y) \mid y < x\}.$$

Conversely, if g is continuous at x, we then have :

$$f(x) \le g(x) = \sup_{y<x} g(y) = \sup_{y<x}\left(\inf_{y<z} f(z)\right) \le \sup_{y<x}\left(\inf_{y<z<x} f(z)\right) \le f(x).$$

Hence $f(x) = g(x)$ and f is right-continuous at x. Moreover, by Proposition 2, there exists an increasing sequence (y_n) in $[\alpha,\beta]$, converging to x, such that f is continuous at each y_n. It follows that :

$$f(x) = g(x) = \sup_n\left(\inf_{y_n<z} f(z)\right) = \sup_n f(y_n) \le \sup_{y<x} f(y).$$

We conclude that $f(x) = \sup_{y<x} f(y)$ and, finally, f is also left-continuous at x.

(2) $\Rightarrow$ (1). If (2) is true, then for every order interval $[a,b]$ in E, each sequence in $[a,b]$ must have a subsequence weakly converging to some point in $[a,b]$, that is $[a,b]$ is weakly compact (by Eberlein-Smulian theorem again [**1**, Th. 10.13]). It follows from Proposition 1 that E has order continuous norm. □

Remark. The function g in the previous proof is not necessarily right-continuous at α or left-continuous at β, as shown by the following example. For $n = 1, 2, \ldots$, define $f_n : [0,1] \to [0,1]$ by $f_n(x) = 0$ if $x \in [0, \frac{1}{n}]$, $f_n(x) = \frac{1}{2}$ if $x \in]\frac{1}{n}, 1 - \frac{1}{n}[$ and $f_n(x) = 1$ if $x \in [1 - \frac{1}{n}, 1]$. In this case, the function g (which coincide with f) is clearly not right-continuous at 0, nor left-continuous at 1.

In order to consider a Helly-Bray theorem in the setting of Banach lattices, we need to be able to integrate (as in [**3**]) some scalar functions with respect to functions with values in a vector lattice. But, since in [**3**], only continuous scalar functions are considered, we need the following lemma.

Lemma. *Let $[\alpha, \beta]$ be a closed interval in $\mathbb{R}$, E a σ-Dedekind complete vector lattice and f a nondecreasing function from $[\alpha,\beta]$ into E. Let also $\alpha = x_0 < x_1 < \ldots < x_p = \beta$, assume that f is order continuous at $x_1, \ldots, x_{p-1}$, and consider a function $g : [\alpha, \beta] \to \mathbb{R}$ such that*

$$\begin{cases} g(x) = \text{ constant } y_j & \text{if } x \in [x_{j-1}, x_j[, \quad (1 \le j \le p); \\ g(\beta) = y_p. \end{cases}$$

Then g is integrable with respect to f and

$$\int_\alpha^\beta g(t)\, df(t) = \sum_{j=1}^p y_j [f(x_j) - f(x_{j-1})].$$

Proof. For simplicity, we only prove that $\int_\alpha^{x_1} g(t)\, df(t) = y_1 [f(x_1) - f(\alpha)]$. Given any partition $\alpha = x'_0 < x'_1 < \ldots < x'_r = x_1$ of $[\alpha, x_1]$ and points $t_i \in [x'_{i-1}, x'_i]$, $(1 \le i \le r)$, we have :

$$\left| \sum_{i=1}^r g(t_i)[f(x'_i) - f(x'_{i-1})] - y_1[f(x_1) - f(\alpha)] \right| =$$

$$|g(t_r) - y_1| \cdot [f(x_1) - f(x'_{r-1})] \le |y_2 - y_1| . [f(x_1) - f(x'_{r-1})].$$

We now choose an increasing sequence (α_n) in $[\alpha, x_1[$, converging to x_1. Since f is order continuous at x_1, we have :

$$f(x_1) = \sup\{f(x) \mid \alpha \le x < x_1\} = \sup_n f(\alpha_n).$$

Then $e_n = |y_2 - y_1| . [f(x_1) - f(\alpha_n)]$ is a nonincreasing sequence in E such that $\inf_n e_n = 0$. On the other hand, letting $\delta_n = x_1 - \alpha_n > 0$, we see that

$$\max_{1 \le i \le r} (x'_i - x'_{i-1}) \le \delta_n \Rightarrow |y_2 - y_1| . [f(x_1) - f(x'_{r-1})] \le e_n.$$

The proof is complete. □

Theorem 2 (Helly-Bray theorem). *Consider a closed interval $[\alpha,\beta]$ in $\mathbb{R}$, a Banach lattice E with order continuous norm and an order interval $[a,b]$ in E. Let (f_n) be a sequence of nondecreasing functions from $[\alpha,\beta]$ into $[a,b]$ and assume there exists a nondecreasing function $f : [\alpha,\beta] \to [a,b]$ such that $\lim_{n\to\infty} f_n(x) = f(x)$ for the weak topology $\sigma(E,E')$ at each continuity point $x \in]\alpha,\beta[$ of f, but also for $x=\alpha$ and $x=\beta$. Then, for each continuous function $g : [\alpha,\beta] \to \mathbb{R}$, we have*

$$\lim_{n\to\infty} \int_\alpha^\beta g(t)\,df_n(t) = \int_\alpha^\beta g(t)\,df(t), \quad \text{for } \sigma(E,E').$$

Proof. By Proposition 2, we know that f has at most countably many points of discontinuity. For $p = 1,2,\ldots$, let $\alpha = x_0^{(p)} < x_1^{(p)} < \ldots < x_{k_p}^{(p)} = \beta$ be points of continuity of f, excepted perhaps α and β, such that $|g(x)-g(y)| \le \frac{1}{p}$ if $x,y \in [x_{j-1}^{(p)}, x_j^{(p)}]$, $(1 \le j \le k_p)$. Define g_p on $[\alpha,\beta]$ by $g_p(x) = g(x_{j-1}^{(p)})$ if $x \in [x_{j-1}^{(p)}, x_j^{(p)}[$, $(1 \le j \le k_p)$ and $g_p(\beta) = g(x_{k_p-1}^{(p)})$. By the above Lemma, we have

$$\int_\alpha^\beta g_p(t)\,df_n(t) = \sum_{j=1}^{k_p} g(x_{j-1}^{(p)})[f_n(x_j^{(p)}) - f_n(x_{j-1}^{(p)})],$$

and this weakly converges in n to

$$\sum_{j=1}^{k_p} g(x_{j-1}^{(p)})[f(x_j^{(p)}) - f(x_{j-1}^{(p)})] = \int_\alpha^\beta g_p(t)\,df(t).$$

On the other hand, it follows from

$$\left|\int_\alpha^\beta (g-g_p)\,df_n\right| \le \frac{1}{p}(b-a) \text{ and } \left|\int_\alpha^\beta (g-g_p)\,df\right| \le \frac{1}{p}(b-a)$$

that

$$\lim_{p\to\infty} \int_\alpha^\beta (g-g_p)\,df_n = \lim_{p\to\infty} \int_\alpha^\beta (g-g_p)\,df = 0,$$

uniformly in n for the norm topology. The result follows. □

Corollary. *With the same assumptions as in Theorem 2, we have also*

$$\lim_{n\to\infty} \int_\alpha^\beta f_n(t)\,dg(t) = \int_\alpha^\beta f(t)\,dg(t), \quad \text{for } \sigma(E,E').$$

Proof. By the formula of integration by parts [**3**], we have

$$\int_\alpha^\beta f_n(t)\,dg(t) = f_n(\beta)g(\beta) - f_n(\alpha)g(\alpha) - \int_\alpha^\beta g(t)\,df_n(t)$$

and similarly for $\int_\alpha^\beta f(t)\,dg(t)$. The result follows from these equalities and Theorem 2. □

2. Applications

As in the classical case, we are now able to prove a moment theorem and to deduce from it a representation theorem for operators on the space of continuous functions on $[0,1]$, by means of a nondecreasing function. Our proofs are easy adaptations of the classical proofs. We include these proofs for sake of completeness.

Let us recall some definitions.

Definition 1. A function f defined on an interval $[\alpha,\beta]$ in $\mathbb{R}$, with values in a Banach lattice E, is said to have an *(o)-bounded variation* if there exists $u \in E$ such that, for any partition $(t_0,\dots,t_n)$ of $[\alpha,\beta]$, the following inequality holds :

$$\sum_{i=1}^{n} |f(t_{i+1}) - f(t_i)| \leq u.$$

Note that a nondecreasing f has certainly an (o)-bounded variation.

Definition 2. A sequence $(a_k)_{k\in\mathbb{N}}$ in a Banach lattice E is a *moment sequence* if there exists a function $f : [0,1] \to E$, with (o)-bounded variation, such that for all k, we have :

$$a_k = \int_0^1 t^k \, df(t).$$

Definition 3. A sequence $(a_k)_{k\in\mathbb{N}}$ in a Banach lattice E is *completely monotone* if for every pair of non-negative integers n, k, we have :

$$\Delta^n a_k = \sum_{j=0}^{n} (-1)^j \binom{n}{j} a_{k+j} \geq 0.$$

Theorem 3 (Moment theorem). *A sequence $(a_k)_{k\in\mathbb{N}}$ in a Banach lattice E with order continuous norm is the moment sequence of a nondecreasing function f if and only if the sequence $(a_k)_{k\in\mathbb{N}}$ is completely monotone.*

Proof. It is clear that if the sequence (a_k) is the moment sequence of a nondecreasing function f then the sequence (a_k) is completely monotone.

Conversely, for $n = 0,1,2,\dots$, let us define a function $f_n : [0,1] \to E$ by the formula :

$$f_n(x) = \sum_{j=0}^{m-1} \binom{n}{j} \Delta^{n-j} a_j \qquad \text{for } x \in \left]\frac{m-1}{n}, \frac{m}{n}\right[$$

$$f_n(\frac{m}{n}) = \sum_{j=0}^{m} \binom{n}{j} \Delta^{n-j} a_j, \; f_n(0) = 0, \; f_n(1) = a_0.$$

The function f_n is nondecreasing and has values in the order interval $[0, a_0]$ of E. If we define the operator Λ on the space of polynomials by

$$\Lambda(\sum_{j=0}^{n} c_j x^j) = \sum_{j=0}^{n} c_j a_j$$

it is clear that the Bernstein polynomials

$$B_{k,n}(x) = \sum_{j=0}^{n} \binom{n}{j} \left(\frac{j}{n}\right)^{k} x^j (1-x)^{n-j}$$

verify

$$\Lambda(B_{n,k}) = \int_0^1 t^k \, df_n(t).$$

Using Theorem 1, we can choose a subsequence $(f_{n_i})_{i\in\mathbb{N}}$ of (f_n) and a nondecreasing function f such that $(f_{n_i}(x))_{i\in\mathbb{N}}$ weakly converges to $f(x)$ at each continuity point $x \in]0,1[$ of f and for $x=0$ and $x=1$. By Theorem 2, for every k, we have :

$$\lim_{i\to\infty} \int_0^1 t^k \, df_{n_i}(t) = \int_0^1 t^k \, df(t), \qquad \text{for } \sigma(E,E').$$

We now show that $\lim_{n\to\infty} \Lambda(B_{k,n}) = a_k$ for the norm topology of E, and the conclusion will follow. By classical algebraic computations, it is easy to show that $a_0 = \Lambda(B_{0,n})$ and that

$$a_k = \sum_{j=k}^{n} \frac{j(j-1)\dots(j-k+1)}{n(n-1)\dots(n-k+1)} \binom{n}{j} \Delta^{n-j} a_j.$$

Consequentely, we obtain :

$$a_k - \Lambda(B_{k,n}) = \sum_{j=k}^{n} \left(\frac{j(j-1)\dots(j-k+1)}{n(n-1)\dots(n-k+1)} - \left(\frac{j}{n}\right)^k \right) \binom{n}{j} \Delta^{n-j} a_j$$
$$- \sum_{j=0}^{k-1} \binom{n}{j} \left(\frac{j}{n}\right)^k \Delta^{n-j} a_j.$$

Let $y = \frac{j}{n}$, and observe that

$$\frac{j(j-1)\dots(j-k+1)}{n(n-1)\dots(n-k+1)} - \left(\frac{j}{n}\right)^k = \prod_{i=0}^{k-1} \frac{ny-i}{n-i} - y^k.$$

It follows that, given $\varepsilon > 0$, there exists n_0 such that

$$\frac{j(j-1)\dots(j-k+1)}{n(n-1)\dots(n-k+1)} - \left(\frac{j}{n}\right)^k \le \varepsilon \quad \text{for } n \ge n_0$$

and

$$\left| \sum_{j=k}^{n} \left(\frac{j(j-1)\dots(j-k+1)}{n(n-1)\dots(n-k+1)} - \left(\frac{j}{n}\right)^k \right) \binom{n}{j} \Delta^{n-j} a_j \right| \le \varepsilon \quad \text{for } n \ge n_0.$$

It is also clear that

$$\left|\sum_{j=0}^{k-1}\binom{n}{j}\left(\frac{j}{n}\right)^k \Delta^{n-j}a_j\right| \leq \left(\frac{k}{n}\right)^k a_0.$$

Now, it is easy to conclude that if n is large enough we have

$$|a_k - \Lambda(B_{k,n})| \leq 2\varepsilon a_0,$$

which proves the theorem. □

Theorem 3 above improves the "Moment theorem" of **[6]** but is only a special case of the "Moment theorems" of [**4**, **5**]. Nevertheless the above proof is much more elementary than those of [**4**, **5**] because we only use an "Helly's theorem" and not a representation theorem of [**2**]. We conclude this paper by showing that the representation theorem of [**2**], in the special setting of Banach lattices with order continuous norm, is a corollary of the above Theorem 3.

Theorem 4 (Representation theorem). *Every positive linear operator L on $C([\alpha,\beta],\mathbb{R})$, with values in a Banach lattice E with order continuous norm, is representable in the form*

$$L(g) = \int_\alpha^\beta g(t)\, df(t),$$

where f is a nondecreasing function from $[\alpha,\beta]$ into E.

Proof. It is clear that it suffices to prove the result for the interval $[0,1]$. The sequence $(a_k)_{k\in\mathbb{N}}$ defined by the formula $a_k = L(t^k)$ is completely monotone. In fact, we have

$$\Delta^n a_k = \sum_{j=0}^{n}\binom{n}{j} a_{k+j} = \sum_{j=0}^{n}\binom{n}{j} L(t^{k+j}) = L(t^k(1-t)^n) \geq 0.$$

By the preceeding theorem, there exists a nondecreasing function f such that

$$L(t^k) = \int_0^1 t^k\, df(t)$$

and, by Weierstrass theorem, this equality extends to every continuous function. We recall that a positive linear mapping from a Banach lattice into a normed vector lattice is automatically continuous ([**1**, Theorem 12.3]). □

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