CONSTRUCTION OF JOINT OBSERVABLE FOR l-RING VALUED MEASURES

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ABSTRACT. In this paper the product measure theorem for l-ring valued measures is proved. The idea of proof is based on a version of Alexandroff theorem. This result is applied to the existence of the joint observable for some types of observable

1. INTRODUCTION

Definition 1.1. An *l-ring* is a system $\mathcal{L} = (L, +, \cdot, \leqq)$ with two binary operations $+, \cdot$ and partial ordering $\leqq$ such that following properties are satisfied:

1. $(L, +)$ is Abelian group;
2. $(L, \leqq)$ is a lattice;
3. $a, b, c \in L; a \leqq b \Rightarrow a + c \leqq b + c$;
4. $\cdot : L^+ \times L^+ \to L^+$ is associative and commutative;
5. there is element $u \in L^+$ with $u \cdot a = a$ for all $a \in L^+$;
6. $a \cdot (b + c) = a \cdot b + a \cdot c$.

Example 1.2. Let $(\Omega, \mathcal{S})$ be a measurable space and $\mathcal{F} = \{f; f : \Omega \to \mathbb{R}, f$ is measurable$\}$. Then $(\mathcal{F}, +, \cdot, \leqq)$ with usual sum, product and partial ordering of functions is an l-ring.

Remark 1.3. We are going to work just with interval $[0, u] \subset \mathcal{L}$.

Remark 1.4. The notion l-ring is a generalization of notion MV-algebra in some case. By the Mundici representation theorem [3] each MV-algebra is isomorphic with an interval $[0, u] \subset G$, where G is a lattice ordered group (l-group). (It is easy to see that the first three points in the Definition 1.1 represent exactly the definition of an l-group.)

2. l-RING VALUED ALEXANDROFF THEOREM

This chapter is a generalization of the paper [1], where the semigroup valued Alexandroff theorem was introduced.

Lemma 2.1. *Let G be an l-group, $\mathcal{D}$ a semiring of subsets of a set, $s(\mathcal{D})$ the generated ring, $\lambda : \mathcal{D} \to G$ an additive mapping. Then there exists exactly one additive extension $\bar{\lambda} : s(\mathcal{D}) \to G$ of λ.*

Proof. The complete proof is available in [1]. □

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In the further Lemma the mapping $\lambda : \mathcal{R} \to G$, where $\mathcal{R}$ be a ring, will be supposed to be monotone, subadditive and compact.

A mapping $\lambda : \mathcal{R} \to G$ is called to be:

(1) *monotone*, if $A \subset B; A, B \in \mathcal{R}$ then $\lambda(A) \leqq \lambda(B)$,

(2) *subadditive*, if $\lambda(A \cup B) \leqq \lambda(A) + \lambda(B)$ for every $A, B \in \mathcal{R}$.

(3) A family $\mathcal{C}$ of subsets of a set is to said to be *compact*, if it is closed under finite intersection and every decreasing sequence of non-empty sets of $\mathcal{C}$ has a non-empty intersection.

(4) A mapping $\lambda : \mathcal{R} \to G$ is said to be *compact*, if there is a compact family $\mathcal{C} \subset \mathcal{R}$ such that to every $E \in \mathcal{R}$ there are $C_n \in \mathcal{C}$ $(n = 1, 2, \dots)$ such that $C_n \subset C_{n+1} \subset E$ $(n = 1, 2, \dots)$ and $\lambda(E \backslash C_n) \searrow 0$.

Recall that l-group G is σ-complete if every upper bounded sequence $(a_i)_i$ has the supremum $\vee a_i$.

Lemma 2.2. *Let G be σ-complete l-group. Let $\lambda : \mathcal{R} \to G$ be monotone, subadditive, compact and $\lambda(\emptyset) = 0$. Then λ is upper continuous in $\emptyset$, i.e.,*

$$(A_n) \subset \mathcal{R}, A_n \searrow \emptyset \Rightarrow \lambda(A_n) \searrow 0.$$

Proof. See [1]. □

Theorem 2.3. *Let $\mathcal{L}$ be a σ-complete l-ring, $\mathcal{R}$ be a ring of subsets of a set, $\lambda : \mathcal{R} \to \mathcal{L}$ be a monotone, additive and compact mapping such that $\lambda(\emptyset) = 0$. Then λ is upper continuous in $\emptyset$.*

Proof. Every σ-complete l-ring is a σ-complete l-group. Hence by Lemma 2.2 the result follows. □

3. σ-ADDITIVE l-RING VALUED MEASURES AND THE PRODUCT MEASURE THEOREM.

Let $\mathcal{R}$ be a ring of subsets of a set. A function $\lambda : \mathcal{R} \to \mathcal{L}$ is said to be $\sigma - additive$ if $\lambda(A) = \vee_{n=1}^{\infty} (\lambda(A_1) + \dots + \lambda(A_n))$, whenever $A \in \mathcal{R}$, $A_n \in \mathcal{R}$ $(n = 1, 2, \dots)$ $A_i \cap A_j = \emptyset$ $(i \neq j)$, $A = \cup_{n=1}^{\infty} A_n$.

Lemma 3.1. *Let G be an l-group, $\mathcal{R}$ be a ring of subsets of a set. Then λ is σ-additive if and only if λ is upper continuous in $\emptyset$.*

Proof. See [1]. □

Theorem 3.2. *Let $\mathcal{L}$ be a σ- complete l-ring, $\lambda : \mathcal{R} \to \mathcal{L}$ be a monotone, additive, compact mapping such that $\lambda(\emptyset) = 0$. Then λ is σ-additive.*

Proof. It follows by Lemma 3.1 and Theorem 2.3. □

Further, let two rings $\mathcal{R}_1$, resp. $\mathcal{R}_2$ of a subset of X_1, resp. X_2 be given. Let $\mathcal{R}_1 \times \mathcal{R}_2$ denotes the family of all sets $A \times B$ $(A \in \mathcal{R}_1, B \in \mathcal{R}_2)$. A function $\lambda : \mathcal{R}_1 \times \mathcal{R}_2 \to \mathcal{L}$ will be called partially additive if

$$\lambda(A \cup B, C) = \lambda(A, C) + \lambda(B, C),$$
$$\lambda(D, E \cup F) = \lambda(D, E) + \lambda(D, F),$$

whenever

$$A, B, D \in \mathcal{R}_1, \quad C, E, F \in \mathcal{R}_2, \quad A \cap B = \emptyset, \quad E \cap F = \emptyset.$$

Lemma 3.2. *Let G be an l-group, $\lambda : \mathcal{R}_1 \times \mathcal{R}_2 \to G$ be a partially additive mapping. Then λ is additive.*

Proof. See [1]. □

Assume for this moment that $\lambda : \mathcal{R} \to G$ is a measure (G is σ-complete l-group). If we want to extend any G-valued measure from a ring to the generated σ-ring, we need a special property of G, so-called *weak σ-distributivity* ([2], [5]). The property is a necessary and sufficient condition for this extension.

Definition 3.3. A σ-complete l-group G is said to be *weakly σ-distributive* if for every bounded sequence $(a_{ij})_{ij} \subset G$ such that $a_{ij} \searrow 0$ ($j \to \infty$, $i = 1, 2, \dots$) it is

$$\bigwedge_{\phi \in \mathbb{N}^{\mathbb{N}}} \bigvee_{i=1}^{\infty} a_{i\phi(i)} = 0.$$

The following Definition comes naturally from the previous.

Definition 3.4. A σ-complete l-ring $\mathcal{L}$ is said to be *weakly σ-distributive*, if σ-complete l-group $\mathcal{L}$ is weakly σ-distributive.

Theorem 3.5 (Product Measure Theorem). *Let $\mathcal{L}$ be a weakly σ-distributive l-ring. If $\mu, \nu : \mathcal{B}(\mathbb{R}) \to [0, u]$ are positive measures (i.e., σ-additive, non-negative and having 0 in $\emptyset$), then there exists exactly one measure $\bar{\lambda} : \mathcal{B}(\mathbb{R}^2) \to [0, u]$ such that $\bar{\lambda}(A \times B) = \mu(A) \cdot \nu(B)$ for every $A, B \in \mathcal{B}(\mathbb{R})$.*

Proof. First define the function $\lambda : \mathcal{D} \to [0, u]$ on $\mathcal{D} = \{[a, b) \times [c, d);\ a, b, c, d \in \mathbb{R}\}$ by the formula

$$\lambda(A \times B) = \mu(A) \cdot \nu(B).$$

Evidently λ is partially additive, hence additive by Lemma 3.2. It can be uniquely extended to an additive mapping $\lambda : s(\mathcal{D}) \to [0, u]$ defined by the formula:

$$\bar{\lambda}\left(\bigcup_{i=1}^{n}(A_i \times B_i)\right) = \sum_{i=1}^{n} \mu(A_i) \cdot \nu(B_i),$$

$(A_i \times B_i) \cap (A_j \times B_j) = \emptyset$ $(i \neq j)$. Moreover, the mapping $\bar{\lambda}$ is monotone, $\bar{\lambda}(\emptyset) = 0$ and compact with respect to $\mathcal{C} = \{C \subset \mathbb{R}^2;\ C \textit{ compact}\}$. Then $\bar{\lambda}$ is σ-additive by Theorem 3.2., hence $\bar{\lambda} : s(\mathcal{D}) \to [0, u]$ is an l-ring valued a measure. By [4] $\bar{\lambda}$ can be uniquely extended to a measure on the generated σ-algebra $\sigma(s(\mathcal{D})) = \mathcal{B}(\mathbb{R}^2)$. □

4. Joint observable for l-ring

Now we are able to construct a joint observable of two observables for a special case of l-ring - weakly σ-distributive l-ring.

Definition 4.1. By an *observable* we shall mean a mapping $x : \mathcal{B}(\mathbb{R}) \to [0, u]$ satisfying the following properties:

1. $x(\mathbb{R}) = u$;
2. If $A, B \in \mathcal{B}(\mathbb{R}), A \cap B = \emptyset$, then $x(A \cup B) = x(A) + x(B)$;
3. If $A_n \in \mathcal{B}(\mathbb{R}) (n = 1, 2, \dots), A_n \nearrow A$, then $x(A_n) \nearrow x(A)$.

In the following example the correspondence between an observable a random variable is shown.

Example 4.2. Let $(\Omega, \mathcal{S})$ be a measurable space, S be a σ-algebra, $\mathcal{F}_{[0,1]} = \{f; f : \Omega \to [0,1], f \text{ is measurable}\}$ and $\xi : \Omega \to \mathbb{R}$ be a random variable. Define $x : \mathcal{B}(\mathbb{R}) \to \mathcal{F}_{[0,1]}$ by the equality $x(A) = \chi_{\xi^{-1}(A)}$. Then x is an observable according to Definition 4.1.

If a binary operation $\cdot$ on $[0, u]$ is given , the joint observable of two observables can be introduced

Definition 4.3. Let $x, y : \mathcal{B}(\mathbb{R}) \to [0, u]$ be observables. The *joint observable* of observables x and y is a mapping $h : \mathcal{B}(\mathbb{R}^2) \to [0, u]$ with the following properties:

1. $h(\mathbb{R}^2) = u$;
2. If $A, B \in \mathcal{B}(\mathbb{R}^2), A \cap B = \emptyset$, then $h(A \cup B) = h(A) + h(B)$;
3. If $A_n \in \mathcal{B}(\mathbb{R}^2)(n = 1, 2, \dots), A_n \nearrow A$, then $h(A_n) \nearrow h(A)$;
4. $h(C \times D) = x(C) \cdot y(D)$ for every $C, D \in \mathcal{B}(\mathbb{R})$.

It is easy to see that the joint observable corresponds to the random vector $T = (\xi, \eta)$.

Example 4.4. Let $x, y : \mathcal{B}(\mathbb{R}) \to \mathcal{F}_{[0,1]}$ be observables

$$x(A) = \chi_{\xi^{-1}(A)} \qquad y(A) = \chi_{\eta^{-1}(A)}$$

where ξ, η are random variables. If $T(\xi, \eta)$ then

$$h(C \times D) = \chi_{\xi^{-1}(C)} \cdot \chi_{\eta^{-1}(D)} = \chi_{T^{-1}(C \times D)}$$

where $T^{-1}(C \times D) = \xi^{-1}(C) \cap \eta^{-1}(D)$.

Now we are able to prove the main result of this paper.

Theorem 4.5. *Let $\mathcal{L}$ be a weakly σ-distributive l-ring, and $x, y : \mathcal{B}(\mathbb{R}) \to [0, u]$ be observables. Then there exists the joint observable of observables x, y.*

Proof. Put $\mu(A) = x(A)$, $\nu(B) = y(B)$ for $\forall\ A, B \in \mathcal{B}(\mathbb{R})$. Then μ, ν are l-ring valued measures. Let $\bar{\lambda}$ be the product measure constructed in Theorem 3.5.

Put for $\forall\ C \in \mathcal{B}(\mathbb{R}^2)$

$$h(C) = \bar{\lambda}(C).$$

Then $h : \mathcal{B}(\mathbb{R}^2) \to [0, u]$. The proof of the properties (1)-(4) is straightforward. □

References

[1] Duchoň, M. - Riečan, B., *On the product of semigroup valued measures*, Tatra Mountains **10** (1997), 17-27.

[2] Fremlin, D. H., *A direct proof of the Matthes-Wright integral extension theorem*, J. London Math. Soc. **11** (1975), 276-284.

[3] Mundici, D., *Interpretation of AFC*-algebras in Lukasiewicz sequential calculus*, J. Funct. Anal. **65** (1986), 15-63.

[4] Riečan, B., *On measures and integrals with values in ordered groups*, Math. Slovaca **33** (1983), 153-163.

[5] Volauf, P., *On various notions of regularity in ordered spaces*, Math. Slovaca **35** (1985), 127-130.

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