A GENERALIZED MOMENT PROBLEM IN VECTOR LATTICES

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ABSTRACT. We present a moment problem in the context of vector lattices. The corresponding generalized moment theorem and its corrolaries are stated and proved.

1. Introduction

If g is a real-valued function of bounded variation on the unit interval I of the real line, the numbers

$$a_k = \int_0^1 t^k dg(t), \qquad k \in N,\ N = \{0, 1, \dots\}$$

are called the moments of g. A sequence of real numbers $(a_n, n \in N)$ is said to give a solution of the moment problem if there exists a function g of bounded variation on I such that

$$a_k = \int_0^1 t^k dg(t)$$

for $k \in N$.

For every sequence of real numbers $a_k, k \in N$ and every pair of non-negative integers n, k, set

$$\Delta^n a_k := \sum_{j=0}^{n} (-1)^j \binom{n}{j} a_{k+j}$$

The sequence $(a_k, k \in N)$ is called completely monotone if $\Delta^n a_k \geq 0$ for all integers $n, k \geq 0$. Hausdorff [4] has shown that for a sequence (a_k) of real numbers to be the moment sequence of some non-decreasing g (this case being of particular interest), it is necessary and sufficient that (a_k) be completely monotone. So the completely monotone sequence gives a non-decreasing solution of the moment problem. The result permits a generalization to the situation where (a_k) is a completely monotone sequence of elements of a (Dedekind) complete vector lattice the definition of completely monotone sequence being the same [3]. In this paper we extend this result for the case that we suppose boundedness of some sums of differences $\Delta^n a_k$, and we make then use the representation of an order bounded linear operator on the space of continuous functions on the interval $[a, b]$ of real line into a (Dedekind) complete vector lattice Y ([1]).

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2. Moment problem theorem

Let Y be a (Dedekind) complete vector lattice. Denote by $L^o(X,Y)$ the vector space of all $o-$bounded operators (that is of order bounded operators) on the normed space X into Y, that is of those that, if $U \in L^o(X,Y)$, then $\{U(x); \|x\| \leq 1\}$ is an $o-$bounded subset of Y. For $U \in L^o(X,Y)$ we put

$$\mid U \mid = \sup\{|U(x)|; \|x\| \leq 1\}$$

If we take into consideration a general form of an order bounded linear operator on the space $C[a,b]$ into Y we can formulate a task in the considerd case as follows: Decide under which conditions there exists a function $g(t) : [a,b] \to Y$ of $o-$bounded variation such that

$$\int_a^b t^n dg(t) = y_n, \quad n = 0, 1, \dots$$

Recall that a function g, defined on $T = [a,b]$ and taking values in Y, is said to be of (o)-bounded variation, if the set of all elements of the form

$$\sum_j |g(t_{j+1}) - g(t_j)|,$$

corresponding to all finite partitions of the interval T, is o-bounded. We shall denote by $(o) - var_{t \in T} g(t)$ the least upper bound of this set.

We shall need the following result [1 , 7.1.5].

The general form of the (o)-bounded linear operator $U : C(T) \to Y$ is given by the formula

$$U(f) = \int_T f(t) dg(t)$$

where $g : T \to Y$ is a function of $(o)-$bounded variation.

Denote by $BV^o(T,Y)$ the vector space of all functions on T with values in Y of the $o-$bounded variation.

Let us recall the following [1]. A function $g : [a,b] \to Y$ is said to be of order bounded variation if the set of all elements of the form

$$\sum_j |g(t_{j+1}) - g(t_j)|$$

corresponding to all finite divisions of the interval $[a,b]$, is order bounded. Denote by $(o) - var_{t \in [a,b]} g(t)$ the supremum of this set. Note that if $g : [a,b] \to Y$ is of order bounded variation then $|g| : [a,b] \to Y$ is also of order bounded variation because $\mid |g(t)| - |g(s)| \mid \leq |g(t) - g(s)|$, $s, t \in [a,b]$. Recall also ([2]) that if g is of order bounded variation and f is a continuous function on $[a,b]$ then both integrals

$$\int_a^b f(t) dg(t) \quad \int_a^b g(t) df(t)$$

exist and the integration by parts formula holds

$$\int_a^b f(t) dg(t) = f(b)g(b) - f(a)g(a) - \int_a^b g(t) df(t)$$

We have also the inequality

$$|\int_a^b g(t)df(t)| \leq \int_a^b |g(t)|d|f(t)|$$

for every continuous function f on $[a,b]$.

If we take into consideration a general form of an order bounded linear operator on the space $C[a,b]$ into Y [1] we can formulate a task in the considered case as follows: Given $y_n \in Y$, decide under which conditions there exists such a function of order bounded variation $g(t) : [a,b] \to Y$ that

$$\int_a^b t^n dg(t) = y_n, \quad n = 0, 1, \ldots \tag{6}$$

We shall derive a concrete result relating to a power moment problem in the interval $[0,1]$.

Theorem. *In order that there exists a function of order bounded variation $g(t)$: $[0,1] \to Y$ such that*

$$\int_0^1 t^n dg(t) = y_n, \quad y_n \in Y, \quad n = 0, 1, \ldots \tag{8}$$

it is necessary and sufficient that there exists a constant element M in Y such that

$$\sum_{k=0}^{n} \binom{n}{k} |\Delta^{n-k} y_k| \leq M, n = 0, 1, \ldots, \tag{9}$$

where $\Delta^m y_k$ denotes the m-th differences for the sequence (y_k) defined inductively by equalities

$$\Delta^{m+1} y_k = \Delta^m y_k - \Delta^m y_{k+1}, \quad \Delta^0 y_k = y_k,$$

$$m = 0, 1, \ldots ; k = 0, 1, \ldots . \tag{10}$$

Proof. **The necessity**. Let the moment problem (8) be solved. Denote by L an order bounded linear operator on the space $C[0,1]$ generated by a function of order bounded variation, $g(t) : [0,1] \to Y$, i.e.

$$L(f) = \int_0^1 f(t)dg(t), \ f \in C([0,1]).$$

Put

$$x_k^{(m)}(t) = t^k(1-t)^m, \quad m, k = 0, 1, \ldots . \tag{11}$$

Since

$$x_k^{(m+1)}(t) = t^k(1-t)^{m+1} = t^k(1-t)^m - t^{k+1}(1-t)^m =$$

$$x_k^{(m)}(t) - x_{k+1}^{(m)}(t),$$

we have

$$L(x_k^{(m+1)}) = L(x_k^{(m)}) - L(x_{k+1}^{(m)}),\ m, k = 0, 1, \dots .$$

Further

$$L(x_k^{(0)}) = y_k.$$

If we take into the consideration (10), we can easily see (by induction) that

$$L(x_k^{(m)}) = \Delta^m y_k, \quad m, k = 0, 1, \dots .$$

From this we deduce that (9) is satisfied, because using the integration by parts formula we have the following.

$$\sum_{k=0}^{n} \binom{n}{k} |\Delta^{n-k} y_k| = \sum_{k=0}^{n} \binom{n}{k} |\int_0^1 x_k^{(n-k)}(t) dg(t)| =$$

$$= \sum_{k=0}^{n} \binom{n}{k} |[x_k^{(n-k)}(t)g(t)]_0^1 - \int_0^1 g(t) d(x_k^{(n-k)}(t))| \leq$$

$$\leq \sum_{k=0}^{n} \binom{n}{k} (|x_k^{(n-k)}(1)g(1)| + |x_k^{(n-k)}(0)g(0)|) +$$

$$+ \sum_{k=0}^{n} \binom{n}{k} \int_0^1 |g(t)| d(x_k^{(n-k)}(t)) = |g(0)| +$$

$$+ |g(1)| + \sum_{k=0}^{n} \binom{n}{k} \int_0^1 |g(t)| d(x_k^{(n-k)}(t)) =$$

$$= |g(0)| + |g(1)| + \int_0^1 |g(t)| d(\sum_{k=0}^{n} \binom{n}{k} x_k^{(n-k)}(t)).$$

Since

$$\sum_{k=0}^{n} \binom{n}{k} x_k^{(n-k)}(t) = \sum_{k=0}^{n} t^k (1-t)^{n-k} = [t + (1-t)]^n = 1$$

we have

$$\sum_{k=0}^{n} \binom{n}{k} |\Delta^{n-k} a_k| \leq |g(0)| + |g(1)| + \int_0^1 |g(t)| d(1) =$$

$$= |g(0)| + |g(1)| \leq 2var_{[0,1]} g = M.$$

The sufficiency. Let L_0 denote the operator defined on the set of functions (x_n), $x_n(t) = t^n, n = 0, 1, \dots$ into Y by formula $L_0(x_n) = y_n, n = 0, 1, \dots$. Extend L to the linear hull of the set (x_n), i. e. to the set of all polynomials. Namely if $x(t) = c_0 + c_1 t + \dots + c_n t^n$, we put

$$L(x) = c_0 y_0 + c_1 a_1 = \dots + c_n y_n.$$

Since the functions $x_n(t)$ are linearly independent the definition of L will be unique. The operator L defined above will be clearly additive and homogeneous. We shall see that condition (9) implies that L is order bounded. Let us note that (even without this condition) the operator L is order bounded on the set P_m of polynomials the degree of which is $\leq m$, because P_m is finite-dimensional space (as coordinates we take coefficients of polynomial), hence the convergence in P_m is coordinatewise.

We have

$$L(x_k^{(s)}) = \Delta^s y_k,\ s, k = 0, 1, \ldots.$$

Take any polynomial $p(t)$. Let the degree of $p(t)$ be m. Form the sequence of corresponding Berstein polynomials (of $p(t)$)

$$p_n(t) = B_n(p; t) = \sum_{k=0}^{n} \binom{n}{k} p(\frac{k}{n}) t^k (1-t)^{n-k}.$$

It is well-known that the degree of the polynomial $p_n(t)$ for any $n = 1, 2, \ldots$ is not greater than m, and since $p_n(t)$ uniformly converges to $p(t)$ for $n \to \infty$, we have (according to remarks above) $L(p_n) \to L(p)$. But

$$|L(p_n)| \leq \sum_{k=0}^{n} \binom{n}{k} |p(\frac{k}{n})| |L(x_n^{(n-k)})| \leq$$

$$\leq ||p|| \sum_{k=0}^{n} \binom{n}{k} |\Delta^{n-k} y_k| \leq ||p|| M.$$

If we take the limit on the left side for $n \to \infty$, we obtain

$$|L(p)| \leq ||p|| M.$$

Since p is an arbitrary polynomial, we have proved that L is order bounded on the space of all polynomials. Now we may extend L to the whole space $C[0,1]$, since polynomials are uniformly dense in $C[0,1]$ ([1],VI.3.3). We may now use for this extension the theorem concerning general form of the order bounded linear operator on the space $C[0,1]$ ([1], VII.1.4) according to which

$$L(f) = \int_0^1 f(t) dg(t), \quad f \in C[0,1],$$

where $g(t) : [0,1] \to Y$ is a function of order bounded variation. In particular

$$L(t^n) = \int_0^1 t^n dg(t), \quad n = 0, 1, \ldots$$

We may further extend L to characteristic functions of all intervals in $[0,1]$. By definition $g(t) = L(c_{[0,t]})$.

Remark. We would like to mention that moment theorem can be applied to obtain a new proof of spectral theorem for Hermitian operators. Cf. [5], where moment theorem is considered in ordered locally convex spaces with some properties.

We may apply the preceeding results, for example, to the sequences in all spaces $l_p, L_p(S,\mu)$, $1 \leq p < \infty$. It is interesting to take x_n, $x_n(t) = t^n$, $n = 0, 1, \dots$ as elements of the given spaces and to take the identity (inclusion) mapping from $C[0,1] \to L_p([0,1])$, $1 \leq p < \infty$. Then x_n give a solution of the moment problem in $L_p([0,1])$. Namely

$$x_n = \int_0^1 s^n dg(s),\ n = 0, 1, \dots$$

where $g(s) = c_{[0,s]}, c_{[0,s]}$ being the characteristic function of the interval $[0, s]$.

On the other hand x_n do not give a solution of the moment problem in $C[0,1]$, i. e. by means of the function g with values in $C[0,1]$, which, of course, is not a (Dedekind) complete vector lattice.

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