APPROXIMATION OF CONTINUOUS T-NORMS BY STRICT T-NORMS WITH SMOOTH GENERATORS

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Abstract. Approximation of a general continuous t-norm by means of strict t-norms was recently shown by Fodor and Jenei [2], see also Nguyen et al. [7] and Kreinovich et al. [5]. We give a modified constructive version of mentioned result connecting advantages of all mentioned approaches. Finally, we give an approximation by means of smoothly generated strict t-norms in a constructive way, modifying the results of Jenei and Pap [3].

1. Introduction

The majority of applications of triangular norms deal with continuous t-norms. Recall that a mapping $\mathbf{T}: [0,1]^2 \to [0,1]$ is called a triangular norm if it is symmetric, associative, monotone and 1 is its neutral element, i.e., $\mathbf{T}(x,1) = x$ for all $x \in [0,1]$. For more details about t-norms we recommend [4,6,8].

From the computational point of view, the simpliest processing is when using the strongest t-norm $\mathbf{T}_M$ (= min) and in the case of continuous Archimedean t-norms. The latter t-norms are generated by means of one-place functions (additive or multiplicative generators). In what follows, we will mostly deal with additive generators , though all results can be rewritten for the case of multiplicative generators straightforwardly.

Theorem 1. [6] A t-norm T is continuous Archimedean t-norm if and only if there is a continuous strictly decreasing mapping $f: [0,1] \to [0,\infty]$ with $f(1) = 0$ so that for all $x, y \in [0,1]$

$$\mathbf{T}(x,y) = f^{-1}(\min (f(0), f(x) + f(y))) . \qquad \blacksquare$$

Recall that the function f from Theorem 1 is called an additive generator of the t-norm **T** and it is unique up to a positive multiplicative constant. A continuous Archimedean t-norm **T** is called strict if and only if any of its additive generators f is unbounded in 0 (i.e., if **T** is isomorphic with the product t-norm $\mathbf{T}_P$). Non-strict continuous Archimedean t-norms are called nilpotent t-norms and they are characterized by $f(0) < \infty$. Any nilpotent t-norm **T** is isomorphic with the Lukasiewicz t-norm $\mathbf{T}_L$, $\mathbf{T}_L(x,y) = \max(0,x+y-1)$. Further recall that if a function $f:[0,1]\to[0,\infty]$ is an additive generator of a continuous Archimedean t-norm **T** then the function $\phi:[0,1]\to[0,1]$ defined by $\phi(x) = \exp(-f(x))$ (i.e., ϕ is a continuous strictly increasing function with $\phi(1) = 1$) is called a multiplicative generator of **T** and

$$\mathbf{T}(x,y) = \phi^{-1}(\max(\phi(0),\phi(x)\phi(y))) \quad .$$

Note that a multiplicative generator of a continuous Archimedean t-norm is unique up to a positive power constant and that $\phi(0) = 0$ corresponds to strict t-norms while $\phi(0) > 0$ corresponds to nilpotent t-norms. Further, there are also non-continuous additive/multiplicative generators of t-norms [9]. If such non-continuous generator is non-continuous also on the half-open interval]0,1] then the resulting t-norm **T** is not continuous. However, throughout this paper we will deal with continuous triangular norms only.

Recently, in several papers the approximation of a general continuous t-norm by means of continuous Archimedean t-norms was discussed [2,3,5,7]. This result is of great importance not only from the theoretical point of view, but it allows to deal in several applications of MV-logics with conjunctions generated by means of either additive or multiplicative generators thus reducing the computational complexity. The aim of this paper is to present a modified and more transparent proof of mentioned result based on the original idea of Nguyen et al. [7]. More, we will give a constructive approximation of a general continuous t-norm by means of t-norms generated by smooth generators, which result was firstly discussed in [3]. The crucial point in the next considerations is the well known characterization of continuous triangular norms as ordinal sums of continuous Archimedean t-norms [4,6,8].

Theorem 2. Let **T** be a continuous t-norm. If all elements from [0,1] are idempotent elements of **T**, i.e., $\mathbf{T}(x,x) = x$ for all $x \in [0,1]$, then $\mathbf{T} = \mathbf{T}_M$. If the only idempotent elements of **T** are the trivial idempotents 1 and 0 then **T** is Archimedean and it can be represented by means of either additive or multiplicative generators. Finally, in all remaining cases there is disjoint system $(]a_i,b_i[;\ i \in \mathcal{I})$ of open subintervals of the unit interval [0,1] and a corresponding system of continuous Archimedean t-norms $(T_i;\ i \in \mathcal{I})$ so that

$$T(x,y) = \begin{cases} a_i+(b_i-a_i)T_i((x-a_i)/(b_i-a_i),(y-a_i)/(b_i-a_i) & \exists i \in \mathcal{I} \text{ s.t. } (x,y)\in[a_i,b_i]^2 \\ \min(x,y) & \text{otherwise} \end{cases} .$$

Then **T** is called an ordinal sum of summands $\langle a_i,b_i,T_i\rangle$, $i \in \mathcal{I}$, and the notation
$T \sim (\langle a_i,b_i,T_i\rangle;\ i \in \mathcal{I})$ is used. ■

Recall that if $T \sim (\langle a_i,b_i,T_i\rangle;\ i \in \mathcal{I})$ is an ordinal sum with representation of a continuous t-norm **T** from Theorem 2 then the union $\bigcup_{i\in\mathcal{I}}]a_i,b_i[$ is the set of all non-idempotent elements of **T**.

2. Uniform approximation of a continuous t-norm by means of generators

Since Dombi [1] it is known, see also [2,4], that the strongest t-norm T_M can be approximated by means of generated t-norms (either strict or nilpotent ones).

Theorem 3. [1] Let f be an additive generator of a continuous Archimedean t-norm **T**. For $\lambda \in]0,\infty[$, define $f_\lambda:[0,1]\to[0,\infty]$ by $f_\lambda(x) = (f(x))^\lambda$. Then also f_λ is an additive generator of a continuous Archimedean t-norm T_λ (which is strict if and only if **T** is strict) for any $\lambda \in]0,\infty[$. More, for all $x,y \in [0,1]$,

$$\lim_{\lambda\to\infty} T_\lambda(x,y) = T_M(x,y) = \min(x,y) .$$ ■

The latter result can be still strengthen. Namely, by the next theorem of Fodor and Jenei [2], the convergence in Theorem 2 is uniform!

Theorem 4. [2] Let a sequence of t-norms $(T_n)_{n=1}^{\infty}$ converges pointwisely to a continuous t-norm **T**. Then this convergence is uniform, i.e., for any $\varepsilon > 0$ there is some $n_\varepsilon \in \mathbf{N}$ such that for all $n \geq n_\varepsilon$ and all $x,\ y \in [0,1]$

$$|T_n(x,y) - T(x,y)| < \varepsilon .$$ ■

Now, we will show that each continuous t-norm T can be approximated with an arbitrary small given accuracy by some strict t-norm.

Theorem 5. Let T be a continuous t-norm and let $\delta \in]0,1[$ be given. Then there exist a strict t-norm $T^{<\delta>}$ which is a δ-approximation of T, i.e., for all x, $y \in [0,1]$ it is

$$|T(x,y) - T^{<\delta>}(x,y)| < \delta . \qquad \square$$

Proof.

1)

As a consequence of Theorem 3, the only idempotent t-norm T_M can be approximated by strict t-norms, i.e., for any $\delta > 0$ there is a strict t-norm T_δ such that for all x, $y \in [0,1]^2$, $|T_M(x,y) - T_\delta(x,y)| < \delta$. Recall that starting from an arbitrary additive generator f of a strict t-norm T, the power $f_\lambda = f^\lambda$, $\lambda \in]0,\infty[$, is again an additive generator of a strict t-norm which we denote T_λ, see also Theorem 3 . Denote

$$F_\delta = \inf\ (f(x-\delta)/f(x);\ x \in [\delta,1]) .$$

Note that $F_\delta > 1$. Then for each $\lambda > 1/\log_2 F_\delta$, T_λ is an appropriate δ-approximation of T_M.

Indeed, let $\delta \le x \le y \le 1$. Then $T_M(x,y) = x$ and

$$x \ge T_\lambda(x,y) = f^{-1}((f^\lambda(x)+f^\lambda(y))^{1/\lambda}) \ge f^{-1}(2^{1/\lambda}f(x)) \ge f^{-1}(F_\delta f(x)) \ge f^{-1}(f(x-\delta)) = x-\delta .$$

Further, if $x < \delta$ and $x \le y$, then $0 \le T_\lambda(x,y) \le x = T_M(x,y) < \delta$.

2)

Let T be a nilpotent t-norm and let f be some of its additive generators. For a given $\delta \in]0,1[$, we define a new additive generator $f_\delta: [0,1] \longrightarrow [0,\infty]$ of a strict t-norm $T_{[\delta]}$ which is the required δ-approximation as follows:

$$f_\delta(x) = \begin{cases} f(\delta)\delta/x & \text{if } x \in [0,\delta] \\ f(x) & \text{otherwise} \end{cases} .$$

Similar arguments as in the first step justify the result.

3)

Let **T** be a continuous t-norm which is an ordinal sum with summands $<\alpha_k,\beta_k,T_k>$, $k \in \mathcal{K}$. For a given $\delta \in]0,1[$, define a continuous t-norm T^δ as an ordinal sum with the same summands as **T** but excluding those summands for which $\beta_k-\alpha_k<\delta$. It is evident that T^δ is a δ-approximation of T. More, t-norm T^δ has finite number of summands only! Let T_δ be a δ-approximation of T_M described in 1) and let $T^\delta \sim (<\alpha_i,\beta_i,T_i>,\ i = 1,\dots,n)$, $\beta_i \le \alpha_{i+1}$, $i = 1,\dots,n-1$. Define a continuous t-norm $T^{[\delta]}$ with the same summands as those of T^δ but possibly adding new ones, or modifying the original ones with nilpotent summand t-norms. We will exploit the following fact : if there is an ordinal sum **T** with summand $<\alpha,\beta,T^*>$ and we replace the summand t-norm T^* by its δ-approximation T^{**} then the new t-norm is a δ-approximation of **T**.

Hence, if $0 < \alpha_1$, we add the summand $<0,\alpha_1,T_\delta>$; if $\beta_i < \alpha_{i+1}$ for some $i \in \{1,\dots,n-1\}$ we add the summand $<\beta_i,\alpha_{i+1},T_\delta>$; if $\beta_n < 1$, we add the summand $<\beta_n,1,T_\delta>$. More, if some of t-norms T_i, $i \in \{1,\dots,n\}$, is a nilpotent one, we replace T_i by its corresponding δ-approximation $(T_i)_{[\delta]}$. Consequently, the t-norm $T^{[\delta]}$ is a δ-approximation of **T** and it is an ordinal sum with finite number of summands $<\alpha_j,\beta_j,T_j>$, $j = 1,\dots,m$, where each t-norm T_j is strict and $\bigcup_{j=1}^{m}[\alpha_j,\beta_j] = [0,1]$.

4)

In the last step we will show that if a continuous t-norm **T** is an ordinal sum with two summands, $T \sim (<0,c,T_1>,<c,1,T_2>)$, where both T_1 and T_2 are strict t-norms, then there exists its δ-approximation $T_{<\delta>}$ which is a strict t-norm. Then, by induction, each t-norm $T^{[\delta]}$ constructed in step 3) can be approximated by a strict δ-approximation.

Let f_i be an additive generator of T_i, $i = 1,2$. For a given δ, let

$$G_\delta = \inf\,(f_1(x-\delta/2)-f_1(x);\ x \in]\delta/2,1]).$$

Define

$$\delta^* = \min\,(1 - f^{-1}(G_\delta),\ \delta/2)\ ,\quad \delta^{**} = (\min\,(\delta,1-c))/2 \quad .$$

Now, we are able to define an additive generator f of the strict t-norm $T_{<\delta>}$ we are looking for,

$$f(x) = \begin{cases} f_2((x-c)/(1-c)) & \text{if } x \in [c+\delta^{**},1] \\ ax^2+bx+c & \text{if } x \in [c-c\delta^*,c+\delta^{**}[\\ kf_1(x/c) & \text{if } x \in [0,c-c\delta^*[\end{cases}\quad ,$$

where $y = ax^2+bx+c$ is the only parabol crossing the points

$(c+\delta^{**}, f(c+\delta^{**}))$, $(c, 2f(c+\delta^{**}))$ and $(c-c\delta^*, 4f(c+\delta^{**}))$, and the choice of k ensures the continuity of f.

We will show that the strict t-norm $\mathbf{T}_{<\delta>}$ generated by f is really a δ-approximation of $\mathbf{T}$.

i) If x, $y < c-c\delta^*$ then $\mathbf{T}(x,y) = \mathbf{T}_{<\delta>}(x,y)$.

ii) If $x < c-c\delta^*$ and $y \geq c-c\delta^*$ then $x \geq \mathbf{T}(x,y) \geq \mathbf{T}(x, c-c\delta^*) = c\mathbf{T}(x/c, 1-\delta^*) \geq c(\max(0, x/c - \delta/2)) > x - \delta$, and similarly $x \geq \mathbf{T}_{<\delta>}(x,y) \geq \mathbf{T}_{<\delta>}(x, c-c\delta^*) = \mathbf{T}(x, c-c\delta^*) > x - \delta$.

iii) If $x \in [c-c\delta^*, c]$ and $y \geq c-c\delta^*$ then $c \geq \mathbf{T}(x,y) \geq \mathbf{T}(c-c\delta^*, c-c\delta^*) \geq \max(0, c-c\delta^* - c\delta/2) > c - \delta$ and similarly $c \geq \mathbf{T}_{<\delta>}(x,y) > c - \delta$.

iv) If $x \in [c, c+\delta^{**}]$ and $y \geq c$ then $c+\delta/2 \geq c+\delta^{**} \geq \mathbf{T}(x,y) \geq c$ and $c+\delta/2 \geq c+\delta^{**} \geq \mathbf{T}_{<\delta>}(x,y) \geq \mathbf{T}_{<\delta>}(c,c) = c-c\delta^* > c-\delta/2$.

v) If x, $y \geq c+\delta^{**}$ and $\mathbf{T}(x,y) < c+\delta^{**}$ then $c < \mathbf{T}(x,y) < c+\delta/2$ and $c+\delta/2 \geq c+\delta^{**} > \mathbf{T}_{<\delta>}(x,y) > \mathbf{T}_{<\delta>}(c,c) = c-c\delta^* > c-\delta/2$.

vi) If x, $y \geq c+\delta^{**}$ and $\mathbf{T}(x,y) \geq c+\delta^{**}$ then $\mathbf{T}(x,y) = \mathbf{T}_{<\delta>}(x,y)$.

We have covered all possible cases for x, $y \in [0,1]$. Combining the steps 1) -4) and replacing the given constant δ by some smaller multiple of δ if necessary, we have just proved the theorem. ■

It is evident that a similar approximation of a continuous t-norm $\mathbf{T}$ by means of nilpotent t-norms is also possible. Indeed, for a given $\delta > 0$, it is enough to find an appropriate nilpotent t-norm $\mathbf{T}^*$ which is $\delta/2$-close to the strict t-norm $\mathbf{T}^{<\delta/2>}$ which is a $\delta/2$-approximation of given continuous t-norm $\mathbf{T}$ constructed as suggested in Theorem 5. Then $\mathbf{T}^*$ is δ-approximation of $\mathbf{T}$. Note that if some function $f: [0,1] \rightarrow [0,\infty]$ is an additive generator of the strict t-norm $\mathbf{T}^{<\delta/2>}$ then it is enough to define a new additive generator $f^*: [0,1] \rightarrow [0,\infty]$ (which will generate $\mathbf{T}^*$) as follows:

$$f^*(x) = \begin{cases} f(\delta/2)(1-x)/(1-\delta/2) & \text{if } x \in [0,\delta/2] \\ f(x) & \text{otherwise} \end{cases} .$$

3. Smoothly generated approximations of continuous t-norms

Recall that by [3] a smoothly generated continuous Archimedean t-norms possesses smooth generators (either additive or multiplicative) only, i.e., all mentioned generators are differentiable functions of all orders $n \in \mathbf{N}$. Based on the results of Theorem 5, we will state a constructive smoothly generated δ-approximation of a given continuous t-norm **T**. Our construction is based on the following lemma.

Lemma 1. Let $h: [0,1] \longrightarrow [0,1]$ be a function defined by

$$h(x) = (1 - \exp(-(1-x)^{-2}))\exp(1 - x^{-2}) \text{ whenever } x \in \,]0,1[$$

and

$$h(0) = 0 \ , \ h(1) = 1 \ .$$

Then the function h is a smooth function on $[0,1]$ and for any $n \in \mathbf{N}$ the n-th derivatives $h^{(n)}(0^+) = h^{(n)}(1^-) = 0$. ■

Theorem 6. Let **T** be a continuous t-norm and let $\delta \in \,]0,1[$ be given. Then there exist a smoothly generated strict t-norm $\mathbf{T}^*$ which is a δ-approximation of **T**. □

Proof.

Let $\mathbf{T}^{<\delta/2>}$ be a strict $\delta/2$-approximation of **T** constructed as shown in Theorem 5 and let ϕ be a multiplicative generator of $\mathbf{T}^{<\delta/2>}$. Then ϕ is a continuous strictly increasing automorphism of the unit interval $[0,1]$ and consequently, both ϕ and its inverse function ϕ^{-1} are uniformly continuous functions on $[0,1]$. Let $\nu > 0$ be a constant such that if for $u, v \in [0,1]$ it is $|u - v| < \nu$ then $|\phi^{-1}(u) - \phi^{-1}(v)| < \delta/4$. Further, let $\gamma = \nu/3$ and let $\beta > 0$ be a constant such that
$|\phi(x) - \phi(y)| < \gamma$ whenever $x, y \in [0,1]$, $|x - y| < \beta$.
Fix an integer $n \geq \max(1/\beta, 4/\delta)$ and put $x_i = i/n$, $y_i = \phi(x_i)$, $i = 0,1, \ldots, n$. Now, we will construct a smooth multiplicative generator ϕ^* by means of the smooth function h from Lemma 1 as follows:

$$\phi^*(x) = y_{i-1}+(y_i-y_{i-1})h((x-x_{i-1})/(x_i-x_{i-1})) \quad \text{whenever } x \in [x_{i-1},x_i]$$

for some $i \in \{1,\ldots,n\}$.

It is easy verification that the function ϕ^* is well defined and smooth. More, both ϕ and ϕ^* are continuous strictly increasing automorphisms of the unit interval crossing the same points (x_i,y_i), $i = 0,1,\ldots,n$. It is then evident that for all $x \in [0,1]$, $|\phi(x) - \phi^*(x)| < \gamma$.

Consequently, for any x, $y \in [0,1]$ it is $|\phi(x)\phi(y) - \phi^*(x)\phi^*(y)| < 2\gamma + \gamma^2 \le \nu$. Further, for any $u \in [0,1]$ it is $|\phi^{-1}(u) - \phi^{*-1}(u)| < 1/n \le \delta/4$. Denote the corresponding strict t-norm as T^*. Then for all x, $y \in [0,1]$ we get

$$|T^*(x,y) - T^{<\delta/2>}(x,y)| = |\phi^{*-1}(\phi^*(x)\phi^*(y)) - \phi^{-1}(\phi(x)\phi(y))|$$
$$\le |\phi^{*-1}(\phi^*(x)\phi^*(y)) - \phi^{-1}(\phi^*(x)\phi^*(y))| + |\phi^{-1}(\phi^*(x)\phi^*(y)) - \phi^{-1}(\phi(x)\phi(y))|$$
$$\le \delta/4 + \delta/4 = \delta/2 .$$

Now, it is evident that $|T^*(x,y) - T(x,y)| < \delta$ for all x, $y \in [0,1]$. ■

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