LINEARITY OF FUZZY FUNCTION DERIVATIVES AND INTEGRALS

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ABSTRACT. A graph of a real fuzzy function can be considered as a collection of nonintersecting graphs of real functions (level functions), connecting values with the same membership degree. It is possible to define a derivative of a fuzzy function in terms of derivatives of its level functions. This derivative at a given point is a fuzzy real number. We show that if we use the same triangular norm for addition of fuzzy functions and addition of their derivatives, then we obtain the usual property $(f+g)' = f' + g'$ while using different triangular norms only one of the inequalities $(f+g)' \leq f' + g'$ or $(f+g)' \geq f' + g'$ holds, depending of the mutual relation of the used triangular norms. Similar considerations are done for integrals of fuzzy functions.

We will deal with a real fuzzy function of a real (crisp) variable, i.e. with a function that assigns a fuzzy number to a real number. First let us consider a fuzzy number as an LR-fuzzy number (for more details see [5]) with both shape functions strictly monotone. The addition of LR-fuzzy numbers based on Zadeh's extension principle is studied in [3] and [4]. We can define *level functions* for a fuzzy function in the following way:

If $\alpha \in (0;1)$, then the level function f_α of a fuzzy function f is a real function for which $f_\alpha(x) = y$ if and only if $f(x)(y) = \alpha$ and y belongs to the decreasing part of $f(x)$ (i.e. it is greater than the peak of $f(x)$). If $\alpha \in (-1,0)$, then $f_\alpha(x) = y$ if and only if $f(x)(y) = -\alpha$ and y belongs to the increasing part of $f(x)$ (i.e. it is less than the peak of $f(x)$). Finally, by f_1 we denote a real function assigning the peak of $f(x)$ to x.

In [1] a derivative of a fuzzy function is defined, using its level functions. We will briefly recall the definition of this derivative:

Suppose the fuzzy function f is defined at a point a and suppose each its level function is differentiable at a. Then its derivative $f'(a)$ is the fuzzy number with the following property: if $\alpha \in (0;1)$, then the α-cut of $f'(a)$ is the interval $(I;S)$, where

$$I = \inf\{f'_\beta(a); |\beta| \geq \alpha\}$$

and

$$S = \sup\{f'_\beta(a); |\beta| \geq \alpha\}.$$

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It can happen that I or S or both have the infinite value, hence the interval $(I;S)$ may be unbounded.

The interval $(I;S)$ is in fact not an α-cut, but a strict α-cut, but as throughout the whole paper we use only this type of cuts, we omit the word "strict" and use just the term "cut".

Given two fuzzy functions f, g, both differentiable at a a question of the linearity arises. In other words we ask whether it holds $(f+g)'(a) = f'(a) + g'(a)$.

We stress that in general the signs of addition in this formula may have different meaning. On the left-hand side we sum two fuzzy functions, on the right-hand side we sum two fuzzy numbers, their derivatives. Both these additions can be based on different triangular norms. Our first theorem shows that in case these t-norms are equal, the linearity follows.

Theorem 1. *If the fuzzy functions f and g have fuzzy derivatives f' and g' at the point a, their sum $f+g$ has the fuzzy derivative $(f+g)'$ at a, and the summing of both $f$and g and their derivatives is based on the same triangular norm T, then $f'(a) + g'(a) = (f+g)'(a)$.*

Proof. Suppose $0 < \alpha < 1$. We will show that the α-cuts of the fuzzy number $(f+g)'(a)$ coincide with the α-cuts of the fuzzy number $f'(a) + g'(a)$.

Let $x \in [f'(a) + g'(a)]_\alpha$. This is equivalent with the inequality

$$[f'(a) + g'(a)](x) > \alpha$$

and from the extension principle we see that this is equivalent with

$$sup\{T(f'(a)(y), g'(a)(z)), x = y + z\} > \alpha.$$

The last inequality holds if and only if there exist $y_0, z_0 \in R$, for which $x = y_0 + z_0$ and $T(f'(a)(y_0), g'(a)(z_0)) > \alpha$.

This is equivalent with the existence of $\beta, \gamma \in (0;1)$ such that y_0 is in the β-cut of $f'(a)$, z_0 is in its γ-cut and $T(\beta, \gamma) > \alpha$. Although y_0, z_0 need not be values of derivatives for some level functions, their presence in the β and γ-cut respectively is equivalent with the existence of $\beta_1, \beta_2, \gamma_1, \gamma_2$ with

$$|\beta_1| > \beta, |\beta_2| > \beta, f'_{\beta_1}(a) \leq y_0 \leq f'_{\beta_2}(a)$$

and

$$|\gamma_1| > \gamma, |\gamma_2| > \gamma, g'_{\gamma_1}(a) \leq z_0 \leq g'_{\gamma_2}(a).$$

For the real functions in the latter inequalities we can use the linear property and obtain that

$$f'_{\beta_1}(a) + g'_{\gamma_1}(a) = (f_{\beta_1}(a) + g_{\gamma_1}(a))' \leq x \leq (f_{\beta_2}(a) + g_{\gamma_2}(a))' = f'_{\beta_2}(a) + g'_{\gamma_2}(a).$$

But as

$$T(|\beta_1|, |\gamma_1|) > \alpha, T(|\beta_2|, |\gamma_2|) > \alpha,$$

the number x is in the α-cut of $(f(a) + g(a))'$.

Hence both α-cuts are equal and the proposition is proved. □

If we do not assume that the same t-norm is used for addition of fuzzy functions and their derivatives, then we obtain the following result:

Proposition 2. *Let T_1, T_2 be triangular norms, let the symbols $+_{T_1}, +_{T_2}$ denote the addition based on these t-norms. If the fuzzy functions f and g have fuzzy derivatives f' and g' at the point a, then $(f +_{T_1} g)'(a) \leq f'(a) +_{T_2} g'(a)$ if and only if $T_1 \leq T_2$.*

Proof. Let $T_1 \leq T_2$. This inequality is equivalent with the statement that $\rho +_{T_1} \sigma \leq \rho +_{T_2} \sigma$ for any pair of fuzzy numbers ρ, σ. Then by the Proposition 1 and this equivalence we have

$$(f +_{T_1} g)'(a) = f'(a) +_{T_1} g'(a) \leq f'(a) +_{T_2} g'(a),$$

and vice versa, which proves our proposition. □

The same approach to the derivative of a fuzzy function can be used also in the case when we consider fuzzy numbers as functions $A : [0;\infty] \to [0;1]$ for which $A(0) = 0, A(\infty) = 1$ and $A(x) = \sup\{A(t); t \in [0;x)\}$. The functions which assign this type of a fuzzy number to a real number are studied in [2]. Here the author defines an integral of such functions, which is an extension to the Lebesgue integral.

The definition of the integral in [2] is based on the isomorphism between the set of all fuzzy numbers and the set of their pseudoinverses. Due to this the linearity and homogeneity of the integral is satisfied only in case when the minimum t-norm is used for the addition of the function as well as of their integrals.

It is possible to utilise the idea of level functions also for these fuzzy functions and to obtain an integral which is connected with the derivative from [1] via the mean theorem of integral calculus.

Let f be a fuzzy function defined on an interval J. Suppose the level functions f_α are integrable on J for each $\alpha \in (0,1]$. Let

$$I = \inf\left\{\int_I f_\alpha(x)dx; \alpha \in (0,1]\right\},$$

$$S = \sup\left\{\int_I f_\alpha(x)dx; \alpha \in (0,1]\right\} = \int_I f_1(x)dx.$$

The integral of a fuzzy function f on the interval J can be defined as a fuzzy real number $i_J(f)$ in the following way:

$$i_J(f)(x) = \begin{cases} 0 & \text{if } x \leq I, \\ \alpha & \text{if } \alpha = \sup\left\{\gamma \in (0,1]; \int_J f_\gamma(t)dt \leq x\right\}, \\ 1 & \text{if } x \geq S \end{cases}.$$

It is easy to show that this integral is linear assuming we use the same t-norm for addition of functions and their integrals. In case of using different t-norms we have the result similar to that of derivatives:

If $T_1 \leq T_2$, then

$$i_J(f +_{T_1} g) \leq i_J f +_{T_2} i_J g$$

and the opposite inequality holds in case when $T_1 \geq T_2$.

The connection with the derivative based on the method used in [1] is shown in the following proposition:

Proposition 3. *Let f be a fuzzy function with integrable level functions on the interval $J = [a, b]$. Let for $x \in J$ $F(x) = i_{[a,x]}(f)$. If f has all its level functions continuous at $x_0 \in J$, then $F'(x_0) = f(x_0)$.*

We note that the derivative of F in the previous proposition is understood in the sense of [1]. The proof follows directly from the definition of the integral i_J.

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