ON A PECULIAR T-NORM

DANA SMUTNÁ

Faculty of Natural Science, Matej Bel University
Department of Mathematics
Tajovského 40
974 01 Banská Bystrica
SLOVAKIA
E-mail:smutna@fhpv.umb.sk

ABSTRACT. An associative commutative monotone and bounded by minimum binary operation on strictly increasing sequences of natural numbers induces a t-norm.
Namely, we can put $(x_n) * (y_n) = (x_n + y_n - n)$. The corresponding t-norm is left continuous and therefore it is applicable in the fuzzy logic. More, it solves an open problem of E. Pap. Several other interesting properties of this t-norm are investigated, including its residual implicator. Finally, a class of t-norms with similar properties is introduced.

1. Introduction

Definition 1. A *triangular norm* (*t-norm* for short) is a binary operation on the unit interval $[0,1]$, i.e., a function $T : [0,1]^2 \to [0,1]$ such that for all $x, y, z \in [0,1]$ the following four axioms are satisfied:
(T1) *Commutativity*

$$T(x,y) = T(y,x),$$

(T2) *Associativity*

$$T(x, T(y,z)) = T(T(x,y),z),$$

(T3) *Monotonicity*

$$T(x,y) \leq T(x,z) \quad \textit{whenever} \quad y \leq z,$$

(T4) *Boundary Condition*

$$T(x,1) = x.$$

For $x \in]0,1]$, we can write

$$x = \sum_{i=1}^{\infty} \frac{1}{2^{x_i}},$$

which is the unique infinite dyadic expansion of x, where $(x_i)_{i\in N}$ is strictly increasing sequence of natural numbers. It is easy to see that each $x \in]0,1]$ is in a one to one correspondence with $(x_i)_{i\in N}$ strictly increasing sequence of natural numbers.

$$x \approx (x_i)_{i\in N}$$

Remark. Let $x \approx (x_i)_{i\in N}$ and $y \approx (y_i)_{i\in N}$. Then $x < y$ if and only if there exists $k \in N$ such that for all $i \in N$, $i < k$, we have $x_i = y_i$ and $x_k > y_k$. □

Some of recently introduced new t-norms based on above described dyadic expansion are recalled in the following example. These t-norms are not continuous and have a dense set of discontinuity points, see Budinčevič and Kurilič [1].

Example. For $(x,y) \in]0,1]^2$ let

$$x = \sum_{i=1}^{\infty} \frac{1}{2^{x_i}} \quad \textit{and} \quad y = \sum_{i=1}^{\infty} \frac{1}{2^{y_i}},$$

be the unique dyadic representations of x and y.
Then the t-norm $T_1 : [0,1]^2 \to [0,1]$ is given by

$$T_1(x,y) = \begin{cases} \sum_{i=1}^{\infty} \frac{1}{2^{x_i+y_i}}, & \text{if } (x,y) \in]0,1[^2, \\ min(x,y), & \text{otherwise,} \end{cases}$$

and the t-norm $T_2 : [0,1]^2 \to [0,1]$ is given by

$$T_2(x,y) = \begin{cases} \sum_{i=1}^{\infty} \frac{1}{2^{x_i \cdot y_i}}, & \text{if } (x,y) \in]0,1[^2, \\ min(x,y), & \text{otherwise.} \end{cases}$$

Both T_1 and T_2 are Archimedean and strictly monotone t-norms, which are neither left nor right continuous. □

We discuss a t-norm recently suggested by Mesiar [4]. Based on the original idea of Budinčevič and Kurilič [1], see also [2], the reals from the half-open interval]0,1] are transformed into the strictly increasing sequences of natural numbers. An associative commutative monotone (and bounded by minimum) binary operation on strictly increasing sequences of natural numbers induces a t-norm, [3], [4]. However, the mentioned binary operation need not be the coordinatewise extension of some given binary operation on natural numbers as proposed in [1].

The usual requirement in fuzzy logic to a t-norm T to model a conjuction is its left-continuity. Then, the implication can be modeled by the corresponding residual operator. Therefore we will investigate t-norms with similar properties as in above example under additional requirement of their left-continuity.

2. New t-norm

Proposition 1. *For* $(x,y) \in\ ]0,1]^2$ *let*

$$x = \sum_{i=1}^{\infty} \frac{1}{2^{x_i}} \quad and \quad y = \sum_{i=1}^{\infty} \frac{1}{2^{y_i}},$$

$$x \approx (x_i)_{i\in N} \quad and \quad y \approx (y_i)_{i\in N}$$

be the unique dyadic representations of x *and* y.
Let $T_* : [0,1]^2 \to [0,1]$ *be given by*

$$T_*(x,y) = \begin{cases} 0, & if\, min(x,y) = 0, \\ \sum\limits_{i=1}^{\infty} \frac{1}{2^{x_i+y_i-i}}, & otherwise. \end{cases}$$

Then T_* *is a strictly monotone t-norm.*

Proof. We define an operation $*$ on strictly monotone sequences of natural numbers $(x_i)_{i\in N} * (y_i)_{i\in N} = (z_i)_{i\in N}$, where $z_n = x_n + y_n - n$. It is obvious that for $x, y > 0$, $T_*(x,y) \approx (x_i)_{i\in N} * (y_i)_{i\in N}$.

The commutativity of T_* is obvious.

For $x, y, z \in]0,1]$, let $x \approx (x_i)_{i\in N}$, $y \approx (y_i)_{i\in N}$, $z \approx (z_i)_{i\in N}$, then

$$[(x_i)_{i\in N} * (y_i)_{i\in N}] * (z_i)_{i\in N} = (x_i + y_i - i)_{i\in N} * (z_i)_{i\in N} = (x_i + y_i - i + z_i - i)_{i\in N} =$$

$$= (x_i + y_i + z_i - i - i)_{i\in N} = (x_i)_{i\in N} * (y_i + z_i - i)_{i\in N} =$$

$$= (x_i)_{i\in N} * [(y_i)_{i\in N} * (z_i)_{i\in N}].$$

If $0 \in \{x,y,z\}$ then clearly $T_*(T_*(x,y),z) = 0 = T_*(x,T_*(y,z))$. Therefore, associativity is satisfied.

Further, $1 = \sum\limits_{i=1}^{\infty} \frac{1}{2^i}$ implies $T_*(x,1) = \sum\limits_{i=1}^{\infty} \frac{1}{2^{x_i+i-i}} = x$ whenever $x \in]0,1]$, and obviously $T_*(0,1) = 0$, showing that 1 is the neutral element of T_*.

Let $y < z$. Then there exists $k \in N$, such that for all $i \in N$, $i < k$, we have $y_i = z_i$ and $y_k > z_k$. Then $x_i + y_i - i = x_i + z_i - i$ for $i \in N, i < k$ and $x_k + y_k - k > x_k + z_k - k$. But it means, that $(x_i)_{i\in N} * (y_i)_{i\in N} < (x_i)_{i\in N} * (z_i)_{i\in N}$. Now it is easy to see that T_* is strictly monotone t-norm, i.e., the cancellation law holds, $T_*(x,y) = T_*(x,z)$ if and only if either $x = 0$ or $y = z$. □

We can deduce that for all $x \in [0,1], n \in N$, T_* possesses the following properties:

(i) $T_*\left(x, \frac{1}{2^m}\right) = x.\frac{1}{2^m}$; $m \in \mathbb{N}$.

Especially, $T_*\left(x, \frac{1}{2}\right) = \frac{1}{2} \cdot x.$

(ii) $T_*\left(x, \frac{1}{2^m} + \frac{1}{2^n}\right) = \frac{1}{2^{n-1}} \cdot x + \frac{2}{2^{x_1}} \cdot \left(\frac{1}{2^m} - \frac{1}{2^n}\right)$; $m, n \in \mathbb{N}$; $m < n$.

Especially, $T_*\left(x, \frac{3}{4}\right) = \frac{1}{2}x + \frac{1}{2} \cdot \frac{1}{2^{x_1}}.$

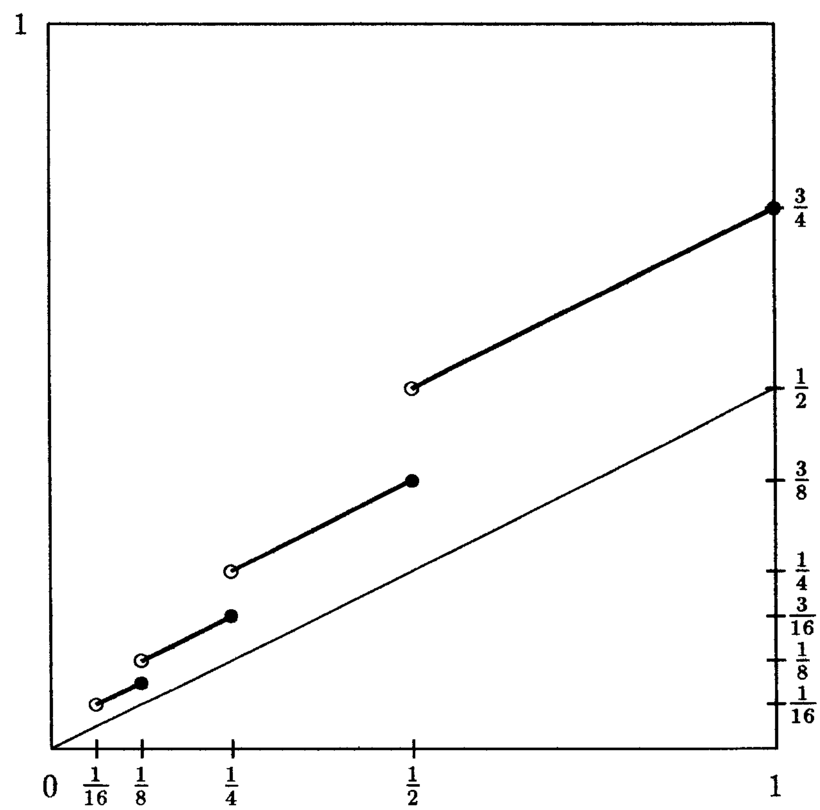

FIGURE 1. Vertical cuts $T_*\left(\frac{1}{2}, x\right)$ and $T_*\left(\frac{3}{4}, x\right)$

(iii) $$T_*\left(x, \frac{1}{2^m} + \frac{1}{2^n} + \frac{1}{2^k}\right) = \frac{1}{2^{k-2}} \cdot x + \frac{1}{2^{x_1}} \cdot \left(\frac{1}{2^{m-1}} - \frac{1}{2^{k-2}}\right) + \frac{1}{2^{x_2}} \cdot \left(\frac{1}{2^{n-2}} - \frac{1}{2^{k-2}}\right); \ m, n, k \in \mathbb{N}; \ m < n < k.$$

Especially, $$T_*\left(x, \frac{7}{8}\right) = \frac{1}{2}x + \frac{1}{2} \cdot \left(\frac{1}{2^{x_1}} + \frac{1}{2^{x_2}}\right).$$

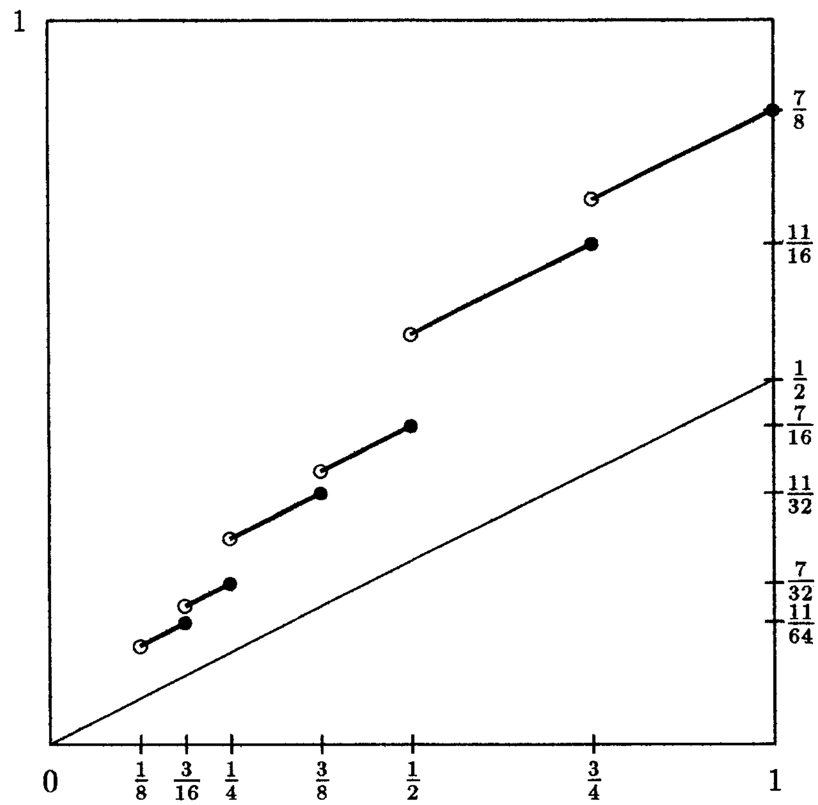

FIGURE 2. Vertical cuts $T_*\left(\frac{1}{2}, x\right)$ and $T_*\left(\frac{7}{8}, x\right)$

3. Some properties of the new t-norm

First, we recall the well-known characterization of left-continuous t-norms.

Proposition 2. *A t-norm T is left-continuous if and only if it is left-continuous in its first component, i.e., if for each $y \in [0,1]$ and for each non - decreasing sequence $(x_n)_{n\in N} \in [0,1]^N$ we have*

$$\sup_{n\in N} T(x_n, y) = T(\sup_{n\in N} x_n, y). \quad \square$$

Now, we will show the left-continuity of the proposed new t-norm.

Proposition 3. *T_* is left continuous.*

Proof. If $y = 0$ or $\sup_{n\in N} x_n = 0$ then for any t-norm T we have

$$\sup_{n\in N} T(x_n, y) = T(\sup_{n\in N} x_n, y) = 0.$$

Let $u \in]0,1], u \approx (u_i)_{i\in N}$ and let $\left(x^{(n)}\right)_{n\in N}$ be a non-decreasing sequence of reals from $]0,1]$ such that

$$\lim_{n\to\infty} x^{(n)} = u \quad \text{and} \quad x^{(n)} \approx (x_i^{(n)})_{i\in N}.$$

Evidently $x^{(n)} < u$ for all $n \in N$. Then there exists $k_n \in N$, such that for all $i \in N, i < k_n$ and we have

$$x_i^{(n)} = u_i \quad \text{and} \quad x_{k_n}^{(n)} > u_{k_n}.$$

Note that

$$\lim_{n\to\infty} x^{(n)} = u \quad \text{implies} \quad \lim_{n\to\infty} k_n = \infty.$$

Now, let $y \in]0,1], y \approx (y_i)_{i\in N}$. Then for any $n \in N$

$$0 < T_*(u,y) - T_*(x^{(n)},y) = \frac{1}{2^{y_{k_n}-k_n}} \cdot \left(\frac{1}{2^{u_{k_n}}} - \frac{1}{2^{x_{k_n}^{(n)}}}\right) + \sum_{i=k_n+1}^{\infty} \left(\frac{1}{2^{u_i+y_i-i}} - \frac{1}{2^{x_i^{(n)}+y_i-i}}\right)$$

and hence

$$0 < T_*(u,y) - T_*(x^{(n)},y) < \frac{1}{2^{u_{k_n}+y_{k_n}-k_n-1}} \le \frac{1}{2^{k_n-1}}.$$

Then

$$\lim_{n\to\infty} \left(T_*(u,y) - T_*(x^{(n)},y)\right) = 0$$

and

$$\sup_{n\in N} T_*(x^{(n)},y) = T_*(u,y) = T_*(\sup_{n\in N} x^{(n)}, y).$$

By Proposition 2, T_* is left-continuous. $\square$

Recall that the left-continuity of T_* allows its application in the framework of fuzzy logic to model a non-continuous fuzzy conjunction.

Proposition 4. *Each point* $(x,y) \in]0,1[^2$, *where at least one coordinate has a finite dyadic representation, is a discontinuity point.*

Proof. Let $u = \frac{1}{2^{m_1}} + \frac{1}{2^{m_2}} + ... + \frac{1}{2^{m_k}}$ be a finite dyadic number from]0,1].
Then $u \approx (m_1, m_2, ..., m_{k-1}, m_k + 1, m_k + 2, m_k + 3...)$.
We can take strictly decreasing sequence $\left(x^{(n)}\right)_{n\in N}$, such that $x^{(n)} \in]0,1], n \in N$, and

$$\lim_{n\to\infty} x^{(n)} = u.$$

Now for arbitrary $y \in]0,1[$, where $y \approx (y_i)_{i\in N}$, it is

$$T_*(u,y) < \frac{1}{2^{m_1+y_1-1}} + ... + \frac{1}{2^{m_{k-1}+y_{k-1}-k+1}} + \frac{1}{2^{m_k+y_k-k}}$$

and

$$T_*(x^{(n)},y) \geq \frac{1}{2^{m_1+y_1-1}} + ... + \frac{1}{2^{m_{k-1}+y_{k-1}-k+1}} + \frac{1}{2^{m_k+y_k-k}}.$$

Then

$$T_*(\inf_{n\in N} x^{(n)}, y) = T_*(u,y) < \inf_{n\in N} T_*(x^{(n)}, y)$$

proving the discontinuity of T_* in point (u,y). □

Remark. This result is also a negative answer to the open problem of E.Pap (Is a strictly monotone t-norm continuous in point $(1,1)$ necessarily always continuous ?), see [5].

As an immediate consequence of Proposition 4 we see the set of all discontinuity points of t-norm T_* is dense in the unit square. Now, we will turn our attention to the Archimedean property. We recall another definition of Archimedean t-norms from [2], [3] which is equivalent with the classical one.

Proposition 5. *A t-norm* T *is Archimedean if and only if for each* $x \in]0,1[$ *we have*

$$\lim_{n\to\infty} \underbrace{T(x, ..., x)}_{\text{n-times}} = 0. \quad \square$$

Now, we stress that T_* is an example of a strictly monotone t-norm, which is not Archimedean.
Indeed, for any $x \in]1 - \frac{1}{2^n}, 1 - \frac{1}{2^{n+1}}]; n \in N$;

$$\lim_{m\to\infty} \underbrace{T_*(x, ..., x)}_{\text{m-times}} = 1 - \frac{1}{2^n},$$

violating the Archimedean property of T_*. □

Let T be a left-continuous t-norm which models a fuzzy conjuction. A standard way how to introduce a fuzzy implication based on T uses the residual operator R_T

$$R_T(x,y) = sup(z \in [0,1]; T(x,z) \leq y).$$

For t-norm T_* we obtain the following residuation:

$$R_{T_*}(x,y) = \begin{cases} \min(1, \frac{y}{x}), & \text{if } x \leq y \quad \text{or} \quad x = \frac{1}{2^n}, \\ \sum\limits_{i=1}^{k} \frac{1}{2^{y_i - x_i + i}} + \frac{1}{2^{y_k - x_k + k}}, & \text{otherwise,} \end{cases}$$

where $(y_i - x_i + i)_{i \in N}$ is an increasing sequence for $i \in \{1, ..., k\}$ and $y_k - x_k + k \geq y_{k+1} - x_{k+1} + k + 1$. Note that $\frac{0}{0} = 1$ by convention.

4. Related t-norms

Another t-norm with similar properties as T_* is T_{**},

$$T_{**}(x,y) = \begin{cases} 0, & \text{if } min(x,y) = 0, \\ \sum\limits_{i=1}^{\infty} \frac{1}{2^{(x_i - i + 1).(y_i - i + 1) + i - 1}}, & \text{otherwise.} \end{cases}$$

Finally, a family of t-norms $(T_k)_{k \in N}$ with similar properties is given by:

$$T_k(x,y) = \sum_{i=1}^{k} \frac{1}{2^{x_i + y_i - i}} + \sum_{i=k+1}^{\infty} \frac{1}{2^{(x_i - i + 1).(y_i - i + 1) + i - 1}}.$$

Remark. Note that t-norms T_* and T_{**} can be understood as limit members of the family $(T_k)_{k \in N}$

$$T_* = T_\infty \quad and \quad T_{**} = T_0.$$

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