The law of large numbers *

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ABSTRACT: A notion of L-R fuzzy number and the law of large numbers are recalled. The LR-fuzzy numbers, for which the law of large numbers hold, are investigated. The case, when the law of large numbers is violated, is studied.

Keywords: fuzzy number, law of large numbers, t-norm

1. Introduction

We recall that fuzzy quantity A is a fuzzy subset of the real line.

Definition 1.

A fuzzy quantity A is a so called *LR-fuzzy number* $A = (a, \alpha, \beta)_{LR}$ if the corresponding membership function satisfies for all $x \in \mathbf{R}$

$$A(x) = \begin{cases} L(\frac{a-x}{\alpha}), & \text{for } a-\alpha \le x \le a, \\ R(\frac{x-a}{\beta}), & \text{for } a \le x \le a+\beta \\ 0, & \text{else} \end{cases},$$

where a is the peak of A; $\alpha > 0$ and $\beta > 0$ is *the left* and *the right spread*, respectively, and L and R are decreasing continuous functions from [0,1] to [0,1] such that $L(0) = R(0) = 1$ and $L(1) = R(1) = 0$. Recall that L and R is called *the left* and *the right shape function*, respectively.

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The addition of fuzzy quantities is based on a given t-norm $\mathbf{T}$, following Zadeh's extension principle, by

$$A \underset{T}{\oplus} B(z) = \sup_{x+y=z} (\mathbf{T}(A(x), B(y))), z \in \mathbf{R}$$

where A, B are given fuzzy quantities. If $\mathbf{T}$ is an Archimedean continuous t-norm with additive generator f then the addition of fuzzy quantities can be expressed as follows

$$A \underset{T}{\oplus} B(z) = f^{(-1)}(\inf_{x+y=z} (f \circ A(x) + f \circ B(y))), \quad z \in \mathbf{R},$$

where $f^{(-1)}$ is the pseudoinverse of f, defined by

$f^{(-1)}(x) = f^{-1}(\min(f(0), x)), \quad x \in [0, +\infty[$.

Let $A_1, \cdots, A_n$ be fuzzy quantities, $n \in \mathbf{N}$ then their $\mathbf{T}$- arithmetic mean M_n is defined as follows $M_n(x) = \frac{1}{n}(A_1 \underset{T}{\oplus} \cdots \underset{T}{\oplus} A_n)(x) := (A_1 \underset{T}{\oplus} \cdots \underset{T}{\oplus} A_n)(nx)$.

2. The law of large numbers for fuzzy numbers

Fullér [2] introduced the law of large numbers for special LR-fuzzy numbers and Hamacher t-norm.

Theorem 1 (The law of large numbers, [2])

Let $A_n = (a_n, \alpha, \alpha)_{LL}, L = 1 - x, n \in \mathbf{N}$, *be symmetric triangular fuzzy numbers and let* $\mathbf{T} \le \mathbf{T}_{H_0}$, *where* $\mathbf{T}_{H_0}(x, y) = \dfrac{xy}{x + y - xy}$ *(whenever* $(x, y) \neq (0, 0)$*) is the Hamacher product. If* $a = \lim_{n \to \infty} \dfrac{a_1 + \cdots + a_n}{n}$ *exists, then*

$$\lim_{n \to \infty} \frac{1}{n}(A_1 \underset{T}{\oplus} \cdots \underset{T}{\oplus} A_n)(z) = \chi_a(z). \qquad (1)$$

However, under some conditions Theorem 1 may fail, see [4] or Example 1.

In the same paper Fullér showed that for $\mathbf{T_M}$ ($\mathbf{T_M}(x,y) = \min(x,y)$) the law of large numbers (1) is violated .

Recently, Hong [5] has shown that to each shape function L there exists a concave shape function L^* such that $L^* \geq L$.

Lemma 1

For any additive generator f and any shape function L there exist a shape function L_f such that $L_f \geq L$ and $f \circ L_f$ is convex.

Proof.

For f corresponding to a nilpotent t-norm **T**, let f^* be the corresponding normed generator. Then $K = 1 - f^* \circ L$ is a shape function and hence there is a shape function $L^* \geq 1 - f^* \circ L$ which is concave. Put $L_f = (f^*)^{-1} \circ (1 - L^*)$. Then $L_f \geq (f^*)^{-1}(1-(1-f^* \circ L)) = L$ and $f \circ L_f = f(0) \cdot f^* \circ (f^*)^{-1} \circ (1 - L^*) = f(0) \cdot (1 - L^*)$ is convex.

If f is a generator of a strict t-norm, then $K = e^{-f \circ L}$ is a shape function. Let $L^* \geq K$ be a concave shape function and put $L_f = f^{-1}(-\ln L^*)$. Then $L_f \geq f^{-1}(-\ln K) = L$ and $f \circ L_f = -\ln L^*$ is convex.

Theorem 2

*Let $A_n = (a, \alpha_n, \beta_n)_{LR}$, $n \in \mathbf{N}$ be LR-fuzzy numbers with bounded spread sequences, i. e., $\alpha_n \leq c, \beta_n \leq c$ for some c > 0. Then the law of large numbers holds with respect to any continuous Archimedean t-norm **T**, i. e.,*

$$\lim_{n\to\infty} \frac{1}{n}\left(A_1 \underset{T}{\oplus} \cdots \underset{T}{\oplus} A_n\right)(z) = \begin{cases} 1 & \text{if } z = a \\ 0 & \text{otherwise} \end{cases}.$$

Proof.

Let $L_f \geq L, R_f \geq R$ be shapes such that the composites $f \circ L_f$ and $f \circ R_f$ are convex (existence of L_f, R_f is ensured by Lemma1). Then $A_n \leq B_n = (a,c,c)_{L_f R_f}, n \in \mathbf{N}$. It is obvious that

$$\frac{1}{n}\left(A_1 \underset{T}{\oplus} \cdots \underset{T}{\oplus} A_n\right)(a) = 1 \text{ for all } n \in \mathbf{N}.$$

Further, for $z \neq a$, say for $z \in]a, a+c]$, we have by [7,8,9]

$$\frac{1}{n}\left(A_1 \underset{T}{\oplus} \cdots \underset{T}{\oplus} A_n\right)(z) \leq \frac{1}{n}\left(B_1 \underset{T}{\oplus} \cdots \underset{T}{\oplus} B_n\right)(z) = f^{-1}\left(\min\left(f(0), n \cdot f \circ R_f\left(\frac{z-a}{c}\right)\right)\right) \to 0.$$

For $z > a+c$, $\frac{1}{n}\left(A_1 \underset{T}{\oplus} \cdots \underset{T}{\oplus} A_n\right)(z) = 0$. The case when $z < a$ is similar.

Theorem 2 can be easily generalized to the case of fuzzy numbers $A_n = (a, \alpha_n, \beta_n)_{L_n R_n}$ such that $\alpha_n \leq c, \beta_n \leq c, L_n \leq L_f, R_n \leq R_f, n \in \mathbf{N}$. However, we cannot drop the requirement of Archimedean property for $\mathbf{T}$ as well as the requirement of equal peaks for $A_n, n \in \mathbf{N}$.

Theorem 3

Let $A_n = (a, \alpha, \beta)_{LR}, n \in \mathbf{N}$ *and let* $\mathbf{T}$ *be a continuous t-norm. We denote* $A(z) = \sup(c \in [0,1];\ c \leq A_1(z), \mathbf{T}(c,c) = c)$. *Then* $\lim_{n \to \infty} \frac{1}{n}\left(A_1 \underset{T}{\oplus} \cdots \underset{T}{\oplus} A_n\right)(z) = A(z)$.

Proof.

Following [1], if $\mathbf{T}(c, c) = c$ and $A_1(z) = c$, then $\frac{1}{n}\left(A_1 \underset{T}{\oplus} \cdots \underset{T}{\oplus} A_n\right)(z) = c = A(z)$ for all $n \in \mathbf{N}$. If $A_1(z)$ is not an idempotent element of $\mathbf{T}$, then it is contained in some interval $]\alpha_k, \beta_k[$ from the ordinal sum decomposition of **T**. Then by [1] and Theorem 2,

$$\lim_{n\to\infty} \frac{1}{n}\left(A_1 \underset{T}{\oplus} \cdots \underset{T}{\oplus} A_n\right)(z) = \alpha_k = A(z).$$

Corollary 1

Let $A_n = (a, \alpha, \beta)_{LR}, n \in \mathbf{N}$ *and let T be a non-Archimedean continuous t-norm. Then the law of large numbers does not hold.*

The following Example of Hong [4] shows that the dropping of the equality of peaks of A_n, $n \in \mathbf{N}$, may violate the law of large numbers.

Example 1

Let $A_n = (a_n, \alpha, \alpha)_{LL}$, $L = 1 - x$, $n \in \mathbf{N}$ and let $\mathbf{T} = \mathbf{T}_{H_0}$, $\mathbf{T}_{H_0}(x, y) = \dfrac{xy}{x + y - xy}$ whenever $(x, y) \neq (0, 0)$, be the Hamacher product. If $b = \lim_{n\to\infty} \sum_{i=1}^{n} a_i$ exists and $0 < |b| < \infty$ then the law of large numbers is violated.

Indeed, since $a = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} a_i = 0$ then by Fullér [2]

$$\lim_{n\to\infty} \frac{1}{n}\left(A_1 \underset{T}{\oplus} \cdots \underset{T}{\oplus} A_n\right)(a) = \lim_{n\to\infty} \frac{1}{n}\left(A_1 \underset{T}{\oplus} \cdots \underset{T}{\oplus} A_n\right)(0) = \lim_{n\to\infty} \frac{n\alpha - |b|}{n\alpha + (n-1)|b|} = \frac{\alpha}{\alpha + |b|} \neq 1.$$

We can generalize Hong's example in the following way.

Theorem 4

Let $A_n = (a_n, \alpha, \beta)_{LR}, n \in \mathbf{N}$, *and let an Archimedean continuous t-norm T with additive generator f, such that* $f \circ L$ *and* $f \circ R$ *are convex, be given. Let* $a = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} a_i$ *and*

$b = \lim_{n\to\infty}\left(\sum_{i=1}^{n} a_i - na\right)$, $|b| \in]0, \infty[$. *Let* $c = \begin{cases} (f \circ L)'(0) & \text{if } b > 0 \\ (f \circ R)'(0) & \text{if } b < 0 \end{cases}$ *be positive. Then*

$$1 > \lim_{n\to\infty}\frac{1}{n}\left(A_1 \underset{T}{\oplus} \cdots \underset{T}{\oplus} A_n\right)(a) = \begin{cases} f^{(-1)}\left(\dfrac{|b|c}{\alpha}\right) & \text{if } b>0 \\ f^{(-1)}\left(\dfrac{|b|c}{\beta}\right) & \text{if } b<0 \end{cases},$$

i. e., the law of large numbers is violated.

Proof.

Suppose that $b > 0$. Then there is some $n_0 \in \mathbf{N}$ such that for all $n \geq n_0$, $\sum_{i=1}^{n} a_i > na$.

Then

$$\frac{1}{n}\left(A_1 \underset{T}{\oplus} \cdots \underset{T}{\oplus} A_n\right)(a) = \left(A_1 \underset{T}{\oplus} \cdots \underset{T}{\oplus} A_n\right)(na) = f^{(-1)}\left(n \cdot f \circ L\left(\frac{\sum_{i=1}^{n} a_i - na}{n\alpha}\right)\right).$$

Now

$$\lim_{n\to\infty}\frac{f \circ L\left(\left(\sum_{i=1}^{n} a_i - na\right)\Big/ n\alpha\right)}{\frac{1}{n}} = (f \circ L)'(0)\cdot\frac{b}{\alpha} = \frac{cb}{\alpha} > 0$$

and consequently $\lim_{n\to\infty} f^{(-1)}\left(n \cdot f \circ L\left(\left(\sum_{i=1}^{n} a_i - na\right)\Big/ n\alpha\right)\right) = f^{(-1)}\left(\frac{cb}{\alpha}\right).$

Note that in Hong's example $a = 0, b = \sum_{n=1}^{\infty} a_n$, $f \circ L(x) = f \circ R(x) = \frac{x}{1-x}$, i. e.,

$c = 1, \alpha = \beta$ and hence $\lim_{n\to\infty}\frac{1}{n}\left(A_1 \underset{T}{\oplus} \cdots \underset{T}{\oplus} A_n\right)(0) = f^{(-1)}\left(\frac{|b|}{\alpha}\right) = \frac{1}{1+\frac{|b|}{\alpha}} < 1.$

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