A NOTE ON SHAPE PRESERVING T–NORM–BASED MULTIPLICATIONS OF FUZZY NUMBERS

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ABSTRACT. The definition of a *log-L-R* fuzzy number is given. Using the results for shape preserving additions of *L-R* fuzzy numbers some results for shape preserving multiplications of *log-L-R* fuzzy numbers are derived. Several illustrating examples are shown.

Key words : fuzzy number, product of fuzzy numbers, triangular norm

1. INTRODUCTION

There are several papers dealing with shape preserving t-norm-based additions of *L-R* fuzzy numbers, e.g., [1, 4, 6, 7 , 9]. In the case of t-norm-based products of *L-R* fuzzy numbers the situation is much more complicated. Recall that an *L-R* fuzzy number is a fuzzy subset of the real line $\mathbb{R}$ whose membership function is given by

$$A(x) = \begin{cases} L\left(\frac{a-x}{\alpha}\right) & \text{for } x \in [a-\alpha, a] \\ R\left(\frac{x-a}{\beta}\right) & \text{for } x \in [a, a+\beta] \\ 0 & \text{otherwise,} \end{cases}$$

where $L, R : [0,1] \to [0,1]$ are shape functions which are continuous, non-increasing, $L(0) = R(0) = 1$, $L(1) = R(1) = 0$ and $a \in \mathbb{R}, \alpha, \beta \in \mathbb{R}^+$.
L-R fuzzy numbers are denoted by $A = (a, \alpha, \beta)_{LR}$.

Fuzzy numbers with shape functions $L(x) = R(x) = 1 - x$, $x \in [0,1]$, are called linear (triangular) and denoted by $A = (a, \alpha, \beta)$.

Let T be a triangular norm (t-norm for short). The product of fuzzy quantities $A, B \in [0,1]^{\mathbb{R}}$ based on a t-norm T is a fuzzy quantity $A \underset{T}{\otimes} B$ which is determined by

$$A \underset{T}{\otimes} B(z) = \sup_{x.y=z} T(A(x), B(y)) \ , \quad z \in \mathbb{R}.$$

For the definition and properties of t-norms we refer to [5].

It is known [11] that in the case of *L-R* fuzzy numbers with supports in $\mathbb{R}^+$, the multiplication based on the weakest t-norm T_W preserves shape functions L and R, and the resulting formula for fuzzy numbers $A_i = (a_i, \alpha_i, \beta_i)_{LR}$ for $i = 1, 2$, is given by

$$A_1 \underset{T_W}{\otimes} A_2 = \left(a_1a_2, a_1a_2 \max(\frac{\alpha_1}{a_1}, \frac{\alpha_2}{a_2}), a_1a_2 \max(\frac{\beta_1}{a_1}, \frac{\beta_2}{a_2})\right)_{LR} .$$

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For fuzzy numbers $A_i = (a_i, \alpha_i, \beta_i)_{LR}$, $i = 1, 2, ..., n$ the T_W-product is given by

$$A_1 \underset{T_W}{\otimes} A_2 \underset{T_W}{\otimes} \dots \underset{T_W}{\otimes} A_n = \left(\prod_{i=1}^{n} a_i, \prod_{i=1}^{n} a_i . \max_{1\le i\le n}(\frac{\alpha_i}{a_i}), \prod_{i=1}^{n} a_i . \max_{1\le i\le n}(\frac{\beta_i}{a_i})\right)_{LR} .$$

Unlike the addition, the multiplication based on the strongest t-norm T_M does not preserve the shapes L, R. Dubois and Prade [3] (see also [11]) derived an approximate formula for the T_M-product of fuzzy numbers $A_i = (a_i, \alpha_i, \beta_i)_{LR}$ for $i = 1, 2$, with supports in $\mathbb{R}^+$, namely,

$$A_1 \underset{T_M}{\otimes} A_2 \approx (a_1 a_2, a_1\alpha_2 + a_2\alpha_1, a_1\beta_2 + a_2\beta_1)_{LR} ,$$

providing that $\frac{a_1}{\alpha_1} + \frac{a_2}{\alpha_2} \gg 1$.

Note that the previous approximate formula for the T_M-product of fuzzy numbers $A_i = (a_i, \alpha_i, \beta_i)_{LR}$ for $i = 1, 2, ..., n$, with supports in $\mathbb{R}^+$, is of the form

$$A_1 \underset{T_M}{\otimes} A_2 \underset{T_M}{\otimes} \dots \underset{T_M}{\otimes} A_n \approx \left(\prod_{i=1}^{n} a_i, \prod_{i=1}^{n} a_i . \left(\sum_{j=1}^{n}(\frac{\alpha_j}{a_j})\right), \prod_{i=1}^{n} a_i . \left(\sum_{j=1}^{n}(\frac{\beta_j}{a_j})\right)\right)_{LR} ,$$

providing that $0 < \frac{\alpha_i}{a_i} \ll 1$, $i = 1, 2, ..., n$.

There is a class of fuzzy numbers for which we can derive some results on t-norm-based multiplications preserving the shapes.

Definition 1. A fuzzy number A is said to be a *log-L-R* fuzzy number if its membership function is given by

$$A(x) = \begin{cases} L\left(\frac{\log a - \log x}{\log \alpha}\right) & \text{for } x \in [\frac{a}{\alpha}, a] \\ R\left(\frac{\log x - \log a}{\log \beta}\right) & \text{for } x \in [a, a.\beta] \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

where $L, R : [0, 1] \to [0, 1]$ are shape functions which are continuous, non-increasing, $L(0) = R(0) = 1$, $L(1) = R(1) = 0$ and a, α, β are real numbers such that $a > 0, \alpha, \beta > 1$.

The *log-L-R* fuzzy numbers will be denoted by $A = (a, \alpha, \beta)_{LR}^{log}$.
If $L(x) = R(x) = 1 - x$, we get *log*-linear fuzzy numbers which will be denoted by $A = (a, \alpha, \beta)^{log}$.

Let $A, B \in [0, 1]^{\mathbb{R}}$ be any two fuzzy quantities. The T-product of A, B, if their supports are subsets of $\mathbb{R}^+$, is given by [11]

$$A \underset{T}{\otimes} B = \exp\left(\log A \underset{T}{\oplus} \log B\right) , \tag{2}$$

since for any fuzzy quantity $C \in [0,1]^{\mathbb{R}}$ it holds

$$(\exp C)(z) = \begin{cases} C(\log z) & \text{for } z > 0 \\ 0 & \text{for } z \leq 0 \end{cases} \tag{3}$$

and

$$(\log C)(z) = C(\exp z)\,, \quad z \in \mathbb{R}. \tag{4}$$

Combining (2) and (3), we obtain

$$A \underset{T}{\otimes} B(z) = \begin{cases} \left(\log A \underset{T}{\oplus} \log B\right)(\log z) & \text{for } z > 0 \\ 0 & \text{for } z \leq 0. \end{cases} \tag{5}$$

Equalities (2) and (5) enable to convert the product of fuzzy quantities into the sum of modified fuzzy quantities.

Lemma 1. *(i) If $A = (a, \alpha, \beta)_{LR}^{log}$ is a log-L-R fuzzy number, then $\log A$ is an L-R-fuzzy number,*

$$\log A = (\log a, \log \alpha, \log \beta)_{LR}.$$

(ii) If A is an L-R fuzzy number, $A = (a, \alpha, \beta)_{LR}$, then $\exp A$ is a log-L-R fuzzy number,

$$\exp A = (a^*, \alpha^*, \beta^*)_{LR}^{log},$$

where $a^* = \exp a$, $\alpha^* = \exp \alpha$, $\beta^* = \exp \beta$.

The proof of the assertion follows directly from the definition of a *log-L-R* fuzzy number and the properties (3) and (4).

Example 1. (1) Let $A = (e^2, e, e)^{log}$, i.e.,

$$A(x) = \begin{cases} \log x - 1 & \text{for } x \in [e, e^2] \\ 3 - \log x & \text{for } x \in [e^2, e^3] \\ 0 & \text{otherwise.} \end{cases}$$

Then $\log A$ is a linear fuzzy number, $\log A = (2, 1, 1)$.

If $B = (e^2, e, e^2)_{LR}^{log}$, where $L(x) = R(x) = 1 - x^2$, then

$$(\log B)(z) = \begin{cases} -z^2 + 4z - 3 & \text{for } z \in [1, 2] \\ -\frac{z^2}{4} + 4 & \text{for } z \in [2, 4] \\ 0 & \text{otherwise,} \end{cases}$$

which means that $\log B = (2, 1, 2)_{LR}$.

2. Limit t-norms-based products of *log-L-R* fuzzy numbers

Let us first consider the product of *log-L-R* fuzzy numbers based on the strongest t-norm T_M, which is defined by

$$T_M(x,y) = \min(x,y), \quad x,y \in [0,1].$$

Proposition 1. *Let A_1, A_2 be log-L-R fuzzy numbers, $A_i = (a_i, \alpha_i, \beta_i)_{LR}^{log}$, $i = 1,2$. Then*

$$A_1 \underset{T_M}{\otimes} A_2 = (a_1a_2, \alpha_1\alpha_2, \beta_1\beta_2)_{LR}^{log}.$$

Proof. By Lemma 1, $\log A_i$ are L-R fuzzy numbers,

$$\log A_i = (\log a_i, \log \alpha_i, \log \beta_i)_{LR}\ , i = 1,2.$$

Therefore

$$\log A_1 \underset{T_M}{\oplus} \log A_2 = (\log a_1 + \log a_2, \log \alpha_1 + \log \alpha_2, \log \beta_1 + \log \beta_2)_{LR},$$

i.e.,

$$\log A_1 \underset{T_M}{\oplus} \log A_2 = (\log a_1a_2, \log \alpha_1\alpha_2, \log \beta_1\beta_2)_{LR}.$$

Using (5), we obtain the claim, which means that the multiplication based on T_M preserves the shapes of *log-L-R* fuzzy numbers.

□

Now, consider the weakest t-norm T_W, which is given by

$$T_W(x,y) = \begin{cases} \min(x,y) & \text{if } \max(x,y) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2. *Let A_1, A_2 be log-L-R fuzzy numbers, $A_i = (a_i, \alpha_i, \beta_i)_{LR}^{log}$, $i = 1,2$. Then*

$$A_1 \underset{T_W}{\otimes} A_2 = (a_1a_2, \max(\alpha_1, \alpha_2), \max(\beta_1\beta_2))_{LR}^{log}.$$

Proof. Since $\log A_i = (\log a_i, \log \alpha_i, \log \beta_i)_{LR}$, $i = 1,2$ are L-R fuzzy numbers, their T_W-sum is given by

$$\log A_1 \underset{T_W}{\oplus} \log A_2 = (\log a_1 + \log a_2, \max(\log \alpha_1, \log \alpha_2), \max(\log \beta_1, \log \beta_2))_{LR},$$

i.e.,

$$\log A_1 \underset{T_W}{\oplus} \log A_2 = (\log a_1a_2, \log(\max(\alpha_1, \alpha_2)), \log(\max(\beta_1, \beta_2)))_{LR}.$$

Combining this with (5), we get the claim. In other words, the multiplication based on the t-norm T_W preserves the shapes of *log-L-R* fuzzy numbers.

□

3. Shape preserving t-norm-based multiplications of log-L-R fuzzy numbers

In the previous section we have shown that both $\underset{T_W}{\otimes}$ and $\underset{T_M}{\otimes}$ preserve the shapes of log-L-R fuzzy numbers.

Using the known results [6, 7] for shape preserving t-norm-based additions of linear fuzzy numbers we can formulate the following sufficient conditions for shape preserving t-norm-based multiplications of log-linear fuzzy numbers also for t-norms different from T_W and T_M.

Proposition 3. *Let* $A_i = (a_i, \alpha_i, \beta_i)^{log}$, $i = 1, 2$ *be log-linear fuzzy numbers.*

(i) If T *is a t-norm with the property* $T_W < T \leq T_L$, *then*

$$A_1 \underset{T}{\otimes} A_2 = (a_1 a_2, \max(\alpha_1, \alpha_2), \max(\beta_1 \beta_2))^{log}. \tag{6}$$

(ii) If T *is a continuous Archimedean t-norm with a concave additive generator then the T-product* $A_1 \underset{T}{\otimes} A_2$ *is also given by (6).*

The proof of this claim is based on the fact that the T-sum of any two linear fuzzy numbers $B = (b, \alpha_B, \beta_B)$ and $C = (c, \alpha_C, \beta_C)$, for T with the property $T_W < T \leq T_L$, is a linear fuzzy number, namely,

$$B \underset{T}{\oplus} C = B \underset{T_W}{\oplus} C = (b + c, \max(\alpha_B, \alpha_C), \max(\beta_B, \beta_C)).$$

The same holds for the T-sum of linear fuzzy numbers for t-norms from (ii), as the concavity of an additive generator of a t-norm T ensures $T \leq T_L$. We omit the details of the proof.

Proposition 4. *Let* $A_i = (a_i, \alpha_i, \beta_i)^{log}$, $i = 1, 2$ *be log-linear fuzzy numbers and let* T_s^Y, $s \in]1, \infty[$ *be the Yager t-norm. Then*

$$A_1 \underset{T_s^Y}{\otimes} A_2 = (a_1 a_2, \alpha, \beta)^{log},$$

where

$$\log^r \alpha = \log^r \alpha_1 + \log^r \alpha_2\ , \quad \log^r \beta = \log^r \beta_1 + \log^r \beta_2 \ \text{ and } \ \frac{1}{r} + \frac{1}{s} = 1 \tag{7}$$

Proof. By [6] , Prop.2, the T_s^Y-sum of linear fuzzy numbers B, C, as above is a linear fuzzy number,

$$B \underset{T_s^Y}{\oplus} C = (b + c, \alpha_*, \beta_*),$$

where

$$\alpha_*^r = \alpha_B^r + \alpha_C^r\ , \quad \beta_*^r = \beta_B^r + \beta_C^r \ \text{ and } \ \frac{1}{r} + \frac{1}{s} = 1.$$

Our claim follows from this result applied to linear fuzzy numbers $\log A_1$, $\log A_2$ and (5).

□

Recall that the Yager t-norms $\{T_s^Y\}_{s\in]0,\infty[}$ are generated by additive generators $f_s(x) = (1-x)^s$. Since the additive generators for $s \in]0,1]$ are concave functions, by [7], the addition based on T_s^Y preserves linearity of fuzzy numbers and the T_s^Y-sum of linear fuzzy numbers is equal to their T_W-sum. From this result and Proposition 3,(ii), it follows that $A_1 \underset{T_s^Y}{\otimes} A_2$ for $s \in]0,1]$, is also given by (6).

Using the results from [9] for shape preserving additions of *L-R* fuzzy numbers we can formulate the following generalized result for *log-L-R* fuzzy numbers.

Proposition 5. *Let T be a t-norm with additive generator f and let $A_i = (a_i, \alpha_i, \beta_i)_{LR}^{log}$, $i = 1,2$ be log-L-R fuzzy numbers. If $f = (L^{-1})^p = (R^{-1})^q$ for some p, $q > 1$, then*

$$A_1 \underset{T}{\otimes} A_2 = (a_1 a_2, \alpha, \beta)_{LR}^{log},$$

where

$$\log^r \alpha = \log^r \alpha_1 + \log^r \alpha_2 \ , \quad \log^s \beta = \log^s \beta_1 + \log^s \beta_2$$

and

$$\frac{1}{p} + \frac{1}{r} = 1 \ , \quad \frac{1}{q} + \frac{1}{s} = 1.$$

Example 2. Let $T = T_L$ be the Lukasiewicz t-norm, $T_L = \max(x + y - 1, 0)$ for $x, y \in [0,1]$. Let $A_1 = (e,e,e)_{LR}^{log}$ and $A_2 = (e^2, e, e^2)_{LR}^{log}$ with $L(x) = R(x) = 1 - x^2$. Then

$$A_1 \underset{T}{\otimes} A_2 = (e^3, e^{\sqrt{2}}, e^{\sqrt{5}})_{LR}^{log}. \tag{8}$$

The additive generator of the Lukasiewicz t-norm is the function $f(x) = 1 - x$, $x \in [0,1]$. It holds that $f(x) = (L^{-1}(x))^2 = (R^{-1}(x))^2$. This means that $p = q = 2$, and therefore $r = s = 2$. Using Proposition 5 we obtain

$$A_1 \underset{T}{\otimes} A_2 = (e^3, \alpha, \beta)_{LR}^{log},$$

where

$$\log \alpha = (\log^2 e + \log^2 e)^{\frac{1}{2}} = \sqrt{2} \ \text{ and } \ \log \beta = (\log^2 e + \log^2 e^2)^{\frac{1}{2}} = \sqrt{5},$$

which gives (8).

4. Conclusions

We have defined a new type of fuzzy quantities, so called *log-L-R* fuzzy numbers, which are in fact exponentials of *L-R* fuzzy numbers.
The multiplication of *log-L-R* fuzzy numbers has similar properties as the addition

of corresponding *L-R* fuzzy numbers. The limit *t*-norm-based multiplications were shown to preserve the shapes of *log-L-R* fuzzy numbers. The linearity of *log-L-R* fuzzy numbers is preserved not only by multiplications based on T_W and T_M but also by multiplications based on any Yager's *t*-norm T_s^Y and on any *t*-norm weaker than T_L.

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