On the pointwise convergence of continuous Archimedean t-norms and the convergence of their generators

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1. Introduction

Continuous Archimedean t-norms are distinguished t-norms which are isomorphic to the product (strict t-norms) or to the Lukasiewicz t-norm (nilpotent t-norms).

Recall that a continuous Archimedean t-norm **T** is fully described by means of the corresponding additive (or multiplicative) generator. Due to the one-to-one correspondence between additive and multiplicative generators we will discuss the case of multiplicative generators only. Recall also that a multiplicative generator ϕ of a continuous Achimedean t-norm **T**, $\phi:[0,1]\to[0,1]$, is a strictly increasing continuous mapping such that $\phi(1) = 1$ and it is unique up to a positive power constant. For strict t-norms $\phi(0) = 0$ while for nilpotent t-norms it is $\phi(0) > 0$. Further, for all x, y $\in$ [0,1] we have

$$T(x,y) = \phi^{-1}(max(\phi(0),\phi(x)\phi(y))) = \phi^{(-1)}(\phi(x)\phi(y)) \quad ,$$

where $\phi^{(-1)}:[0,1]\to[0,1]$ is a pseudo-inverse of ϕ defined by $\phi^{(-1)}(t) = \phi^{-1}(max(\phi(0),x))$. For more details we refer to [3,4,6].

In [5], we have introduced an open problem whether the pointwise convergence of continuous Archimedean t-norms to a limit continuous Archimedean t-norm **T** is equivalent with the pointwise convergence of an appropriate sequence of corresponding generators to a generator of **T**. This problem was positively solved by Jenei [2] by means of additive generators (only the convergence of generators in point 0 may be violated). Note, however, that the proof of sufficiency was omitted in [2]. For the sake of completeness, we include this proof showing that the pointwise convergence of generators to a generator results to the pointwise convergence of corresponding Archimedean t-norms. Further, in the present paper we give a shorter solution of the above convergence problem by means of multiplicative generators.

2. Convergence of generators results the convergence of t-norms

There are several families of continuous Archimedean t-norms converging to limit t-norm members. For any family of continuous Archimedean t-norms $\{T_n\}$ we can find an appropriate family of corresponding multiplicative generators $\{\phi_n\}$ such that the limit $\phi = \mathit{lim}\ \phi_n$ exists. So, e.g., taking sufficiently large power constants, we can force the resulting limit ϕ to be 0 on interval $[0,1[$ while $\phi(1) = 1$. However, ϕ is not a multiplicative generator of a t-norm. On the other hand, if ϕ is a multiplicative generator of a t-norm, we are interested whether also the family $\{T_n\}$ converges to a limit t-norm **T** generated by ϕ. Note that the well known example of Frank's family of t-norms where strict t-norms converge to a nilpotent limit member shows that the limit property for generators in point 0 may be violated without influencing the limit property of t-norms (the limit of strict t-norm generators in point 0 is always 0 while the generator of nilpotent t-norm should be positive in 0).

Theorem 1. Let $\{T_n;\ n \in \mathbf{N}\}$ be a family of continuous Archimedean t-norms with respective multiplicative generators $\{\phi_n;\ n \in \mathbf{N}\}$ and let the pointwise limit on $]0,1]$ interval $\mathit{lim}\ \phi_n = \phi:]0,1] \to]0,1]$ be a strictly increasing continuous mapping, i.e., ϕ continuously extended to the whole unit interval $[0,1]$ is a multiplicative generator of some continuous Archimedean t-norm **T**. Then $\mathbf{T} = \mathit{lim}\ \mathbf{T}_n$ is the pointwise limit of the given family of t-norms.

Proof. Note first that $\phi_n(1) = 1$ for all $n \in \mathbf{N}$ and consequently $\phi(1) = 1$. Further, $\lim \phi_n = \phi$ implies $\mathit{lim}\ \phi_n^{(-1)} = \phi^{(-1)}$. Indeed, if for example $\mathit{limsup}\ \phi_n^{(-1)}(x) > \phi^{(-1)}(x)$ for some $x \in]0,1[$ (note that in 0 all pseudo-inverses of multiplicative generators have the value 0 while in 1 they have the value 1) then there is a positive constant ε and an infinite subsequence $\{n_k\}$ such that $\phi_{n_k}^{(-1)}(x) > \phi^{(-1)}(x) + \varepsilon > 0$ for all n_k. However, then the positivity of $\phi_{n_k}^{(-1)}(x)$ leads to

$$x = \phi_{n_k}(\phi_{n_k}^{(-1)}(x)) > \phi_{n_k}(\phi^{(-1)}(x) + \varepsilon) \longrightarrow \phi(\phi^{(-1)}(x) + \varepsilon) > x \quad ,$$

what is a contradiction. More, the convergence of pseudo-inverses is a uniform convergence (as the pointwise convergence of continuous mappings to a continuous limit on a compact set), i.e., for any $\varepsilon > 0$ we can find an $n_0 \in \mathbf{N}$ such that for all $x \in [0,1]$ and all $n \geq n_0$ it is

$$|\phi^{(-1)}(x) - \phi_n^{(-1)}(x)| < \varepsilon .$$

Further, all mentioned pseudo-inverses are uniformly continuous (as continuous functions on a compact set) and hence there exists some positive constant δ such that

$$|\phi^{(-1)}(t) - \phi^{(-1)}(z)| < \varepsilon$$

whenever $t, z \in [0,1]$ and $|t - z| < \delta$.

Now, take arbitrary two elements $x, y \in [0,1]$. If at least one of these elements is 0, then $\mathbf{T}_n(x,y) = 0 = \mathbf{T}(x,y)$ for all $n \in \mathbf{N}$. Suppose that both x and y are positive. Due to the pointwise convergence of generators there is some $m \geq n_0$ such that for all $n \geq m$ we have

$$|\phi_n(x) + \phi_n(y) - \phi(x) - \phi(y)| < \delta .$$

Then for all $n \geq m$ we have

$$\begin{aligned} |\mathbf{T}_n(x,y) - \mathbf{T}(x,y)| &= |\phi_n^{(-1)}(\phi_n(x) + \phi_n(y)) - \phi^{(-1)}(\phi(x) + \phi(y))| \\ &\leq |\phi_n^{(-1)}(\phi_n(x) + \phi_n(y)) - \phi^{(-1)}(\phi_n(x) + \phi_n(y))| \\ &\quad + |\phi^{(-1)}(\phi_n(x) + \phi_n(y)) - \phi^{(-1)}(\phi(x) + \phi(y))| \\ &< 2\varepsilon \quad . \end{aligned}$$

Since ε can be chosen arbitrarily small , $\mathit{lim}\ \mathbf{T}_n = \mathbf{T}$. ∎

3. Convergence of t-norms results the convergence of generators

Our proof is essentially based on the next convergence theorem which is due to Fodor and Jenei [1].

Theorem 2. Let $\{\mathbf{T}_n\}$ be a sequence of t-norms such that the pointwise limit $\mathbf{T} = \mathit{lim}\ \mathbf{T}_n$ is a continuous t-norm. Then the latest convergence is a uniform one, i.e., for arbitrary $\varepsilon > 0$ there exists $n_0 \in \mathbf{N}$ such that for all $n \geq n_0$, $|\mathbf{T}(x,y) - \mathbf{T}_n(x,y)| < \varepsilon$ for all $x, y \in [0,1]$. ∎

Each t-norm is a binary operation on the unit interval [0,1]. Due to the associativity, there is unique extension of each given t-norm to an n-ary operation on [0,1]. Based on Theorem 2, we have the following convergence theorem.

Theorem 3. Let $\{T_n\}$ be a sequence of t-norms such that the pointwise limit $\mathbf{T} = \lim \mathbf{T}_n$ is a continuous t-norm. For a given $k \in \mathbf{N}$, let $\{a_{n,i}\}$ be infinite convergent sequences of elements from [0,1] with corresponding limits a_i , $i = 1,\dots,k$. Then

$$\lim \mathbf{T}_n(a_{n,1},\dots,a_{n,k}) = \mathbf{T}(a_1,\dots,a_k) \quad .$$

Proof. The proof for $k = 2$ follows from Theorem 2 applying the uniform convergence of $\{\mathbf{T}_n\}$ to $\mathbf{T}$. For $k > 2$ it is enough to apply the method of mathematical induction. ∎

Now, we are able to prove the main result of this paper.

Theorem 4. Let $\mathbf{T}_n$, $n \in \mathbf{N}$, and $\mathbf{T}$ be given continuous Archimedean t-norms and let the pointwise limit $\lim \mathbf{T}_n = \mathbf{T}$. Let ϕ_n, $n \in \mathbf{N}$, and ϕ be the corresponding multiplicative generators with prescribed value $\phi_n(0.5) = \phi(0.5) = 0.5$, $n \in \mathbf{N}$. Then

$$\lim \phi_n(x) = \phi(x) \quad \text{for all } x \in]0,1] \; .$$

Proof. The proof is divided into several steps.

i) It is evident that $\phi_n^{(-1)}(0.5) = \phi^{(-1)}(0.5) = 0.5$ for all $n \in \mathbf{N}$.
The convergence of $\{\mathbf{T}_n\}$ to $\mathbf{T}$ ensures the convergence

$$\lim \mathbf{T}_n(0.5,\dots,0.5) = \mathbf{T}(0.5,\dots,0.5)$$

for any k-tuple $(0.5,\dots,0.5)$, $k \geq 2$ (for $k = 2$ this is just the convergence of $\{\mathbf{T}_n\}$ to $\mathbf{T}$, for $k > 2$ it is enough to apply Theorem 3 as many times as necessary). Consequently,

$$\lim \phi_n^{(-1)}(\phi_n(0.5)^k) = \lim \phi_n^{(-1)}(2^{-k}) = \phi^{(-1)}(2^{-k}) = \phi^{(-1)}(\phi(0.5)^k).$$

ii) For any fixed $k \in \mathbf{N}$, put $a_{n,k} = \phi_n^{-1}(2^{-1/k})$ and $a_k = \phi^{-1}(2^{-1/k})$.

We will show that $\lim \phi_n^{-1}(2^{-1/k}) = \phi^{-1}(2^{-1/k})$. Suppose the contrary and let, e.g., $\limsup a_{n,k} = b > a_k + \varepsilon$ for some $\varepsilon > 0$. Then there is an infinite subsequence $\{a_{n,k_m}\ ;\ m \in \mathbf{N}\}$ converging to b and due to Theorem 3,

$$0.5 = \lim_m \phi^{(-1)}_{n,k_m}(\phi_{n,k_m}(0.5)) = \lim_m \phi^{(-1)}_{n,k_m}(\phi_{n,k_m}(a_{n,k_m})^k)$$

$$= \lim_m T_{n_m}(\underbrace{a_{n,k_m},\ldots,a_{n,k_m}}_{k\text{-times}})$$

$$= \underbrace{T(b,\ldots,b)}_{k\text{-times}} = \phi^{(-1)}(\phi(b)^k) > \phi^{(-1)}(\phi(a_k)^k) = 0.5 \quad ,$$

what is a contradiction. The case when *liminf* $a_{n,k} < a_k - \varepsilon$ for some positive ε is similar.

iii) Combining i), ii) and Theorem 3, we have for any positive rational number r (which can be written as a ratio r = m/k of two numbers m, k $\in$ **N**) the following convergence:

$$\lim \phi^{(-1)}_n(2^{-r}) = \phi^{(-1)}(2^{-r}) \quad .$$

Taking into account the continuity of all pseudo-inverses $\phi^{(-1)}_n$, $n \in \mathbf{N}$, and $\phi^{(-1)}$ as well as the fact that $\phi^{(-1)}_n(1) = \phi^{(-1)}(1) = 1$ and $\phi^{(-1)}_n(0) = \phi^{(-1)}(0) = 0$, the convergence

$$\lim \phi^{(-1)}_n(x) = \phi^{(-1)}(x) \quad .$$

is true for all $x \in [0,1]$.

iv) The convergence of pseudo-inverses $\lim \phi^{(-1)}_n = \phi^{(-1)}$ ensures the desired convergence of multiplicative generators on $]0,1]$. Indeed, let that convergence be violated in some point $x \in\ ]0,1[$ (the convergence in point x = 1 is obvious) and let, e.g., *liminf* $\phi_n(x) < \phi(x)$. Then there is a positive constant ε and an infinite subsequence $\{n_k\}$ so that $0 < \phi_{n_k}(x) < \phi(x) - \varepsilon$ (in the first inequality we need the positivity of x!) and consequently

$$x = \phi^{(-1)}_{n_k}(\phi_{n_k}(x)) < \phi^{(-1)}_{n_k}(\phi(x) - \varepsilon) \longrightarrow \phi^{(-1)}(\phi(x) - \varepsilon) < x \ ,$$

what is a contradiction (note that for x = 0, the fact that $\phi^{(-1)}(t) = 0$ for all $t \le \phi(0)$ may destroy the previous contradiction). The remaining case when *limsup* $\phi_n(x) > \phi(x)$ can be treated similarly. ∎

4. Conclusions

In the light of Theorems 1 and 4, the limit properties of continuous Archimedean t-norms can be investigated by means of the corresponding multiplicative (additive) generators. More, we can approximate (uniformly, see Theorem 2) strict t-norms by nilpotent t-norms and vice-versa.

Proposition 1. Let **T** be a given nilpotent t-norm. For any $\varepsilon > 0$ there is a strict t-norm $\mathbf{T}_\varepsilon$ such that for all $x, y \in [0,1]$.

$$|\mathbf{T}(x,y) - \mathbf{T}_\varepsilon(x,y)| < \varepsilon .$$

More, the system $\{\mathbf{T}_\varepsilon\}$ converges uniformly to **T**.

Proof. It is enough to suppose $\varepsilon < 1$. Let ϕ be a multiplicative generator of **T**. For given $\varepsilon \in]0,1[$, define a multiplicative generator ϕ_ε of a strict t-norm $\mathbf{T}_\varepsilon$ as follows:

$$\phi_\varepsilon(x) = \begin{cases} \phi(\varepsilon)x & \text{if } x \le \varepsilon \\ \phi(x) & \text{otherwise} \end{cases} .$$

Now the result is evident. ■

Proposition 2. Let **T** be a given strict t-norm. For any $\varepsilon > 0$ there is a nilpotent t-norm $\mathbf{T}_\varepsilon$ such that for all $x, y \in [0,1]$.

$$|\mathbf{T}(x,y) - \mathbf{T}_\varepsilon(x,y)| < \varepsilon .$$

More, the system $\{\mathbf{T}_\varepsilon\}$ converges uniformly to **T**.

Proof. We can use similar argumentation as in Proposition 1 putting

$$\phi_\varepsilon(x) = \begin{cases} (\phi(\varepsilon)+\phi(x))/2 & \text{if } x \le \varepsilon \\ \phi(x) & \text{otherwise} \end{cases} .$$ ■

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