A NOTE ON GENERATORS OF T-NORMS

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ABSTRACT. Construction of t-norms by means of additive generators is discussed. Examples of additive generators of t-norms whose ranges are not relatively closed under the addition are given. The case of left-continuous additive generators of t-norms is completelly analysed.

Key words: additive generator, pseudo-inverse, triangular norm.

We know that if a function f is an additive generator of a two-place function T and the range of function f is relatively closed under addition then the related function T is a t-norm. Some situations when a function f is an additive generator of a t-norm and the range of function f is not relatively closed under the addition are considered. First we recall several known basic concepts. We know that a function $T\colon [0,1]^2 \to [0,1]$ which is associative, commutative, non-decreasing and fulfils the boundary condition, i.e, $T(x,1) = x$ for all $x \in [0,1]$ is called a triangular norm (a t-norm for short).
Let $f\colon [0,1] \to [0,\infty]$ be a non-increasing function. Then function $f^{(-1)}\colon [0,\infty] \to [0,1]$ defined by

$$f^{(-1)}(y) = \sup\{x \in [0,1] \,|\, f(x) > y\}$$

is called the pseudo-inverse of function f. Note that $\sup \emptyset = 0$. This function $f^{(-1)}$ is non-increasing function.

Definition 1. [2] Let $f\colon [0,1] \to [0,\infty]$ be a non-increasing function.
The range of function f is relatively closed under the addition if and only if for all $x, y \in [0,1]$ we have $f(x) + f(y) \in Range(f)$ or $f(x) + f(y) \geq \lim_{x \to 0+} f(x)$.

Proposition 1. *[2] Let $f\colon [0,1] \to [0,\infty]$ be a strictly decreasing function, $f(1) = 0$ and let $f^{(-1)}$ be the pseudo-inverse of function f. Then the function $T\colon [0,1]^2 \to [0,1]$ defined by formula*

$$T(x,y) = f^{(-1)}(f(x) + f(y)) \tag{1}$$

is commutative, non-decreasing and fulfils the boundary condition.

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Definition 2. [1] Let $f\colon [0,1] \to [0,\infty]$ be a strictly decreasing function, $f(1) = 0$ and let the function $T\colon [0,1]^2 \to [0,1]$ be given by means of (1). Then the function f is called a conjunctive additive generator of a function T.

The function defined by (1) is not necessarily associative. Next Theorem 1 describes a construction of a t-norm T based on a special conjunctive additive generator f.

Theorem 1. *[2] Let $f\colon [0,1] \to [0,\infty]$ be a strictly decreasing function whose range is relatively closed under the addition and which satisfies $f(1) = 0$ and let function $f^{(-1)}\colon [0,\infty] \to [0,1]$ be defined by $f^{(-1)}(y) = \sup\{x \in [0,1] \,|\, f(x) > y\}$. Then the function $T\colon [0,1]^2 \to [0,1]$ defined by*

$$T(x,y) = f^{(-1)}(f(x) + f(y))$$

is a t-norm.

Now we will study one real situation when *Range*(f) is not relatively closed under the addition.

Example 1. Let $0 \le \delta < \infty$ and $\frac{1}{2} \le v \le \frac{1}{2} + \delta$. Define a function $f_{\delta,v}\colon [0,1] \to [0,\infty]$ by the following formula

$$f_{\delta,v}(x) = \begin{cases} 1 - x + \delta & \text{if } 0 \le x < \frac{1}{2} \\ v & \text{if } x = \frac{1}{2} \\ 1 - x & \text{if } \frac{1}{2} < x \le 1. \end{cases}$$

Note that $\delta = \lim\limits_{x \to \frac{1}{2}^-} f(x) - \lim\limits_{x \to \frac{1}{2}^+} f(x)$, $v = f_{\delta,v}(\frac{1}{2})$ and $f_{\delta,v}$ is a conjunctive additive generator. Instead of $T_{\delta,v}$ which is corresponding two-place function generated by function $f_{\delta,v}$ we will write only T.

i) Let $\delta = 0$, then $v = \frac{1}{2}$. The function $f_{0,\frac{1}{2}}(x) = 1 - x$ for all $x \in [0,1]$ is continuous and it is an additive generator of Lukasiewicz t-norm.

ii) Let $0 < \delta < \frac{1}{2}$ ($\frac{1}{2} \le v \le \frac{1}{2} + \delta$). In this case the corresponding generated function T is not a t-norm. Indeed, if $x = y = \frac{3}{4} - \frac{\delta}{2}$ and $z = \frac{3}{4} + \frac{\delta}{2}$, we have

$$T(x, T(y,z)) = T\left(x, \frac{1}{2}\right) = \frac{3}{4} + \frac{1}{2}\delta - v <$$

$$< min\left(\frac{1}{2}, \frac{3}{4} + \frac{3}{2}\delta - v\right) = T\left(\frac{1}{2}, z\right) = T(T(x,y), z)$$

violating the associativity of T.

iii) Let $\delta \ge \frac{1}{2}$ ($\frac{1}{2} \le v \le \frac{1}{2} + \delta$). Corresponding functions T are the same on set $([0,1]/\frac{1}{2}) \times ([0,1]/\frac{1}{2})$ for every $f_{\delta,v}$.

A. Let $\delta \geq v$ and $v \leq \frac{1}{2}(\delta + \frac{1}{2})$, then the generated function $T\colon [0,1]^2 \to [0,1]$ is defined by

$$T(x,y) = \begin{cases} 0 & \text{if } x+y<1 \\ \frac{1}{2} & \text{if } (x,y) \in M \\ x+y-1 & \text{otherwise.} \end{cases}$$

where $M = \{(x,y) \in [0,1]^2 \mid \frac{1}{2} \leq x, y \leq 1 \text{ and } x+y \leq \frac{3}{2}\}$.
Due to Proposition 1 it is enough to show the associativity of T. Using the definition of T, we obtain

$$T((x,y),z) = \begin{cases} 0 & \text{if } x+y<1 \\ 0 & \text{if } x+y \geq 1; (x,y) \in M \text{ and } 0 \leq z < \frac{1}{2} \\ \frac{1}{2} & \text{if } x+y \geq 1; (x,y) \in M \text{ and } \frac{1}{2} \leq z \leq 1 \\ 0 & \text{if } x+y \geq 1; (x,y) \notin M \text{ and } x+y+z<2 \\ \frac{1}{2} & \text{if } x+y \geq 1; (x,y) \notin M; x+y+z \geq 2 \text{ and} \\ & (x+y-1,z) \in M \\ x+y+z-2 & \text{if } x+y \geq 1; (x,y) \notin M; x+y+z \geq 2 \text{ and} \\ & (x+y-1,z) \notin M. \end{cases}$$

We can see that

$$C_1 = \{(x,y,z) \in [0,1]^3 \mid x+y \geq 1;\ (x,y) \in M \text{ and } \frac{1}{2} \leq z \leq 1\} =$$

$$= \{(x,y,z) \in [0,1]^3 \mid \frac{1}{2} \leq x,y,z \leq 1;\ x+y \leq \frac{3}{2}\}$$

and

$$C_2 = \{(x,y,z) \in [0,1]^3 \mid x+y \geq 1;\ (x,y) \notin M;\ x+y+z \geq 2 \text{ and } (x+y-1,z) \in M\} =$$

$$= \{(x,y,z) \in [0,1]^3 \mid \frac{1}{2} \leq x,y,z \leq 1;\ \frac{3}{2} < x+y \text{ and } x+y+z \leq \frac{5}{2}\}$$

and thus

$$C = C_1 \cup C_2 = \{(x,y,z) \in [0,1]^3 \mid \frac{1}{2} \leq x,y,z \leq 1 \text{ and } x+y+z \leq \frac{5}{2}\}.$$

If $(x,y,z) \in C$, then $T((x,y),z) = \frac{1}{2}$. Consequently, we obtain the following expresion of T:

$$T((x,y),z) = \begin{cases} \frac{1}{2} & \text{if } (x,y,z) \in C \\ 0 & \text{if } (x,y,z) \notin C \text{ and } x+y+z<2 \\ x+y+z-2 & \text{if } (x,y,z) \notin C \text{ and } x+y+z \geq 2. \end{cases}$$

Analogically, by the definition of T, we have

$$T(x,(y,z)) = \begin{cases} 0 & \text{if } y+z<1 \\ 0 & \text{if } y+z\ge 1; (y,z)\in M \text{ and } 0\le x<\frac{1}{2} \\ \frac{1}{2} & \text{if } y+z\ge 1; (y,z)\in M \text{ and } \frac{1}{2}\le x\le 1 \\ 0 & y+z\ge 1; (y,z)\notin M \text{ and } x+y+z<2 \\ \frac{1}{2} & \text{if } y+z\ge 1; (y,z)\notin M; x+y+z\ge 2 \text{ and} \\ & (x,y+z-1)\in M \\ x+y+z-2 & \text{if } y+z\ge 1; (y,z)\notin M; x+y+z\ge 2 \text{ and} \\ & (x,y+z-1)\notin M. \end{cases}$$

From

$$D_1 = \{(x,y,z)\in[0,1]^3 \mid y+z\ge 1;\ (y,z)\in M \text{ and } \frac{1}{2}\le x\le 1\} =$$
$$= \{(x,y,z)\in[0,1]^3 \mid \frac{1}{2}\le x,y,z\le 1 \text{ and } y+z\le\frac{3}{2}\}$$

and

$$D_2 = \{(x,y,z)\in[0,1]^3 \mid y+z\ge 1;\ (y,z)\notin M;\ x+y+z\ge 2 \text{ and } (x,y+z-1)\in M\} =$$
$$\{(x,y,z)\in[0,1]^3 \mid \frac{1}{2}\le x,y,z\le 1;\ \frac{3}{2}<y+z \text{ and } x+y+z\le\frac{5}{2}\}$$

we immediately have

$$D = D_1\cup D_2 = \{(x,y,z)\in[0,1]^3 \mid \frac{1}{2}\le x,y,z\le 1 \text{ and } x+y+z\le\frac{5}{2}\}.$$

If $(x,y,z)\in D$, then $T(x,(y,z))=\frac{1}{2}$. It is obvious that $C=D$. Thus, we obtain

$$T(x,(y,z)) = \begin{cases} \frac{1}{2} & \text{if } (x,y,z)\in C \\ 0 & \text{if } (x,y,z)\notin C \text{ and } x+y+z<2 \\ x+y+z-2 & \text{if } (x,y,z)\notin C \text{ and } x+y+z\ge 2. \end{cases}$$

We have just shown that

$$T((x,y),z) = T(x,(y,z)) \text{ for every } x,y,z\in[0,1]$$

proving the associativity of T. Consequently, the function T is a t-norm.

B. Let $\delta\ge v$ and $v>\frac{1}{2}(\delta+\frac{1}{2})$. We prove that the generated function T is not associative in this case. Indeed, if $x=y=\frac{3}{4}$ and $z=\frac{1}{2}$, then

$$T(T(x,y),z) = T\left(T\left(\frac{3}{4},\frac{3}{4}\right),\frac{1}{2}\right) = T\left(\frac{1}{2},\frac{1}{2}\right) = f^{(-1)}(2v) <$$

$$< \frac{1}{2} = T\left(\frac{3}{4}, \frac{1}{2}\right) = T\left(\frac{3}{4}, T\left(\frac{3}{4}, \frac{1}{2}\right)\right) = T(x, T(y, z)),$$

i.e., the function T is not associative.

C. Let $\delta < v$. We prove that the corresponding generated function T is not associative. Denote $p = v - \delta$. If we take $x = y = \frac{1}{2} + \frac{1}{3}p$ and $z = \frac{1}{2} + \frac{2}{3}p$ than we have

$$T(T(x,y),z) = T\left(\frac{1}{2}, z\right) = \frac{1}{2} - \frac{1}{3}p > \frac{1}{2} - \frac{2}{3}p = T\left(x, \frac{1}{2}\right) = T(x, T(y,z))$$

We can summmarize that the function $f_{\delta,v}$ is an additive generator of a t-norm T if and only if $\frac{1}{2} \leq \delta$ and $\frac{1}{2} \leq v \leq \frac{1}{2}(\frac{1}{2} + \delta)$ or $\delta = 0$ and $v = \frac{1}{2}$. More, the function $f_{\delta,v}$ is an additive generator of the same t-norm when $\frac{1}{2} \leq \delta$ and $\frac{1}{2} \leq v \leq \frac{1}{2}(\frac{1}{2} + \delta)$. Note that $f_{\delta,v}$ is not a multiple of f_{δ^*,v^*} whenever $(\delta, v) \neq (\delta^*, v^*)$, though the resulting t-norm T is always the same. Point 0.5 is an idempotent element of T and consequently T is not an Archimedean t-norm.

Proposition 2. *[2] Let a function $f\colon [0,1] \to [0,\infty]$ be a conjunctive additive generator of the function T and let f be left-continuous at point 1.*
Then the function f is continuous on interval $(0,1]$ if and only if $Range(f)$ is relatively closed under the addition.

Proof. We prove only assertion that if function f is not continuous on interval (0,1], then Ran(f) is not relatively closed under the addition since the opposite is obvious.
Let f fulfils all assumptions of this proposition and let f be discontinuous at point $a \in (0,1)$. Denote

(1) $$\epsilon = f(a) - \lim_{t \to a+} f(t) > 0.$$

As far as f is strictly increasing and $\lim_{t \to 1^-} f(t) = 0$, there exist a point $x \in (a,1)$ such that

(2) $$0 < f(x) < \epsilon.$$

Similarly we can choose a point $y \in (a,1)$ such that

(3) $$\lim_{t \to a+} f(f) - f(x) < f(y) < \lim_{t \to a+} f(t).$$

From (1),(2) and (3) we obtain

$$\lim_{t \to a+} f(t) < f(x) + f(y) < f(a).$$

Moreover, the function f is left-continuous, strictly decreasing and $0 < a$, thus we have

$$f(a) = \lim_{t \to a^-} f(t) < \lim_{t \to 0+} f(t).$$

We have just proved that $f(x) + f(y) \notin Ran(f)$ and $f(x) + f(y) < \lim_{t \to 0+} f(t)$.

Now, we are able to show the next result.

Theorem 2. *[6] Let $f\colon [0,1] \to [0,\infty]$ be a left-continuous conjunctive additive generator of the function T. If the function T is associative then the function f is continuous on interval (0,1].*

For the proof and more details see [6].
Let $f\colon [0,1] \to [0,\infty]$ be a left-continuous conjunctive additive generator of the function T. If f is continuous on interval $(0,1]$ then due to Proposition 2, *Range*(f) is relatively closed under the addition and due to Theorem 1, a corresponding function T is a t-norm. These results can be formulated into the next Theorem 3.

Theorem 3. *[6] Let $f\colon [0,1] \to [0,\infty]$ be a left-continuous conjunctive additive generator of the function T.*
Then the function T is a t-norm if and only if a function f is continuous on interval (0,1].

By the one-to-one correspondence between additive and multiplicative generators all presented results can be rewritten for multiplicative generators, too.

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REFERENCES

[1] Klement,E.P.,Mesiar,R.,Pap,E., *Additive generators of t-norms which are not necessarily continuous*, EUFIT'96,Vol 1, Aachen, 1996, 70-73.
[2] Klement,E.P.,Mesiar,R.,Pap,E., *Triangular norms*, (to appear).
[3] Ling,C.M., *Representation of associative functions*, Publ. Math., Debrecen 12, 1965, 189-212.
[4] Mesiar,R., *Generated conjunctors and related operators in MV-logic for AI-applications*, (to appear).
[5] Schweizer,B. and Sklar,A., *Probabilistic metric spaces*, North-Holland, New York, 1983.
[6] Viceník,P., *Triangular norms and additive generators*, (in preparation).