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The Application of the Fuzzy-scenario Optimisation Approach to Immunisation Bond portfolio

1. Introduction

The area of bond portfolio management is typical of the problems that are amenable to solution via such mathematical programming techniques as linear and quadratic, integer, chance-constrained programming. The solution and applied methodology is influenced mainly by future interest rate structures and liabilities obligation estimation. Shapes of interest rate curves are apparent on Fig.1. There is several basic strategies in bond portfolio decision making.

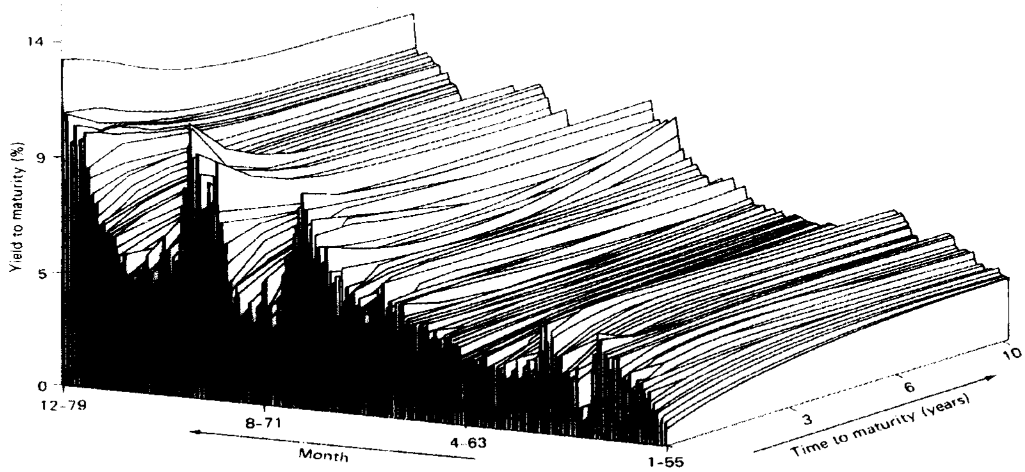


Fig. 1. The development of the interest rates structure USA economy [Haugen93]

Method of matching cash-flow liabilities and assets at any date is entitled the **dedicated bond** approach. This method is hedged against shape shifts of yield curve but this one is most conservative and also much expensive.

The second approach to construction of bond portfolios is depicted as the **immunised bond** method. This one is hedged against the parallel yield curve shifts. There is used concept of the duration and convexity. The duration of bond is a measure of its sensitivity to changes in interest rates. Consider a schedule of cash-flow (cf_t) at dates (t) discounted an annual yield (y), price of bond (PB). Present value (pv) of the schedule is defined as:

$$pv = \sum_t cf_t \cdot (1+y)^{-t} \tag{1}$$

and Macullay duration (d) is defined as:

$$\frac{\partial PB}{\partial y} \equiv d = pv^{-1} \cdot \sum_t t \cdot cf_t \cdot (1+y_t)^{-t} \qquad (2)$$

A portfolio is immunised if the duration, the present value of assets is equal to that of liabilities. The development of the concept of the duration and its adaptation by practitioners as a useful tool is well surveyed in (see [Kaufman82]).

The most yield curve shifts are however non-parallel types. Hence, duration concept does not eliminate other interest rate risk associated with changes in the shape of the yield curve such as flatting or steeping. The **scenario optimisation** method is an approach to optimisation under risk in which the one could be represented by discrete scenarios and different shape yield curves might be considered and modelled.

In described modelling methods the problems are usually assumed that there is perfect or stochastic (a scenario approach) knowledge about the future interest structures and liabilities obligation. A typical feature of the problem is however a vagueness which could be modelled by fuzzy apparatus and expressed linguistically. It concerns mostly of interest rates and liabilities stream.

2. Description of the scenario optimisation model

Scenario optimisation model could be considered to be stochastic methodology and consists of the three basic parts.

Objective function as a global optimisation criterion; balance of cash input output, assets and liabilities; duration strategy and other limits of percentage of portfolio parts. Objective functions may include: cost of the portfolio, average yield, duration weighted-average, terminal market value of own capital and so forth.

Balance of cash inputs and outputs at considered period is the basic assumption of the model. Requirement of the surplus of market value of own capital under new portfolio solution and original portfolio at considered dates is a crucial element of the problem.

Scenarios of yield curves and liabilities stream are basic factors influencing solution of portfolio structure. Matching of duration assets and liabilities is another mean of immunisation of the bond portfolio. Diversification of the bonds types allows to diminish the portfolio risk.

Problem of scenario optimisation could be formulated as follows (variables are depicted by capital letters):

$$\text{maximum} \qquad \sum_j BS_j \cdot ps_j - \sum_k BB_k \cdot pb_k \qquad (3)$$

subject to:

$$\sum_j a_{j,t} \cdot (bh_j - BS_j) + \sum_k b_{k,t} \cdot BB_k + Z_{t-1}(1+z_{t-1}) + S_t = L_t + S_{t-1}(1+s_{t-1}) + Z_t, \qquad t=1..m \qquad (4)$$

$$\sum_j ps_{j,i,t} \cdot (bh_j - BS_j) + \sum_k pb_{k,i,t} \cdot BB_k + NP_t \geq \sum_j ps_{j,i,t} \cdot BH_{j,i,t} + OP_t, \qquad t=1..m,\ i=1..n, \tag{5}$$

$$\text{where } NP_t = Z_t + Z_{t-1}(1+z_{t-1}) - [S_t + S_{t-1}(1+s_{t-1})],$$
$$OP_t = Z'_t + Z'_{t-1}(1+z'_{t-1}) - [S'_t + S'_{t-1}(1+s'_{t-1})],$$

$$\sum_j d_j \cdot ps_j \cdot (bh_j - BS_j) + \sum_k d_k \cdot pb_k \cdot BB_k \leq d\max, \tag{6}$$

$$\sum_j (bh_j - BS_j) \geq 0. \tag{7}$$

The objective function (3) maximise surplus of cash output and input from sold and bought bonds in the decision (trade) date.

The equality of cash-flow input and output at scenario dates is expressed by constraint (4), simultaneously borrowing and reinvestment (lending) of financial sources is allowed.

The market value surplus of the own capital of new portfolio over original one for various scenarios and dates of forecast is modelled by constrain (5).

Constraint (6) limits the maximum average duration of the new bond portfolio. The decision variables and parameters are as follows:

BS_j, BS_k, bh_j,	...par amount of bond sold , bought or held in currency units,
$ps_j, (pb_k)$	...market values (market price plus accrued interest) of bonds sold (bought) per one currency unit,
$a_{j,t}, (b_{k,t})$	...cash-flow (par value, coupon) currency unit due to bond j, (k) at date t,
$Z_t, (S_t)$	...reinvested (borrowed) amount of cash-flow,
$z_t, (s_t)$	...annual reinvested (borrowed) rate,
$NP_t, (OP_t)$	...other assets and liabilities under the new (old) portfolio decision except bonds,
L_t	...payment of obligation portfolio,
$d_j, (d\max)$	...bond duration (maximum portfolio duration),
j,k	...indexes for securities bonds held (sold) in portfolio,
t	...dates of obligations,
′	...superscript for other assets and liabilities under the original portfolio decision except bonds.

It is apparent the problem of bond portfolio optimisation might be considered to be a fuzzy programming problem.

3. Methodology of fuzzy apparatus implementation

Fuzzy programming problem is complicated one and many approaches were described in the literature. The most complex one is the problem under constraints, right hand side fuzzy coefficients and objective function fuzzy coefficients. The general formulation of the problem is as follows (fuzzy elements are depicted by tilde):

$$\max \ \tilde{z}(\tilde{c}, x)$$

$$\text{subject to:} \quad \tilde{A}x \overset{\tilde{R}}{\leq} \tilde{b}$$

It is apparent the types of fuzzy programming problems are determined by following elements: (1) type of fuzzy set, (2) binary fuzzy relation, (3) operations of fuzzy sets, (4) fuzzy objective function.

The applied type of the fuzzy programming problem is characterised by (1) linear T-number fuzzy set, (2) fuzzy approximate operation, (3) binary relation $\tilde{R}_{\varepsilon,\delta}$ type, (4) fuzzy utility type of fuzzy objective function.

Type of fuzzy set

There is many types of fuzzy sets, interpretation and calculation aspects show T-number is eligible for financial applications.

T - number fulfils normality, convexity and continuity characteristics and is defined as follows:

$\tilde{s} = (s^L, s^U, s^\alpha, s^\beta)_{\varphi,\psi}$, where $s^L \leq s^U$; $s^\alpha, s^\beta \geq 0$; $s^L, s^U, s^\alpha, s^\beta \in E^1$.

Functions $\varphi(x)$, $\psi(x)$ are real continuous ones, $\varphi(x)$ is function nondecreasing for $x \in [s^L - s^\alpha; s^L]$; and $\varphi(x) \in [0;1]$, $\psi(x)$ is nonincreasing function for $x \in ['s^U; s^U + s^\beta]$ and $\psi(x) \in ['0;1]$.

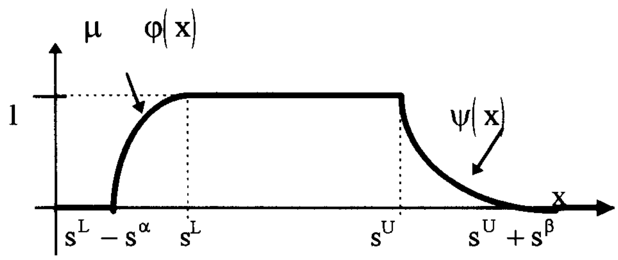


Fig.1: Membership function for the T - number

The membership function for **T-number** is defined as follows:

$$\tilde{s}^T \equiv \mu_{\tilde{s}^T}(x) = \begin{cases} 0 & \text{for} \quad x \leq s^L - s^\alpha \\ \varphi(x) & \text{for} \quad s^L - s^\alpha < x < s^L \\ 1 & \text{for} \quad s^L \leq x \leq s^U \\ \psi(x) & \text{for} \quad s^U < x < s^U + s^\beta \\ 0 & \text{for} \quad x \geq s^U + s^\beta \end{cases}$$

Definition of a **linear T-number**, being a special T-number case of a linear form φ and ψ :

$$\varphi(x) = \frac{x - (s^L - s^\alpha)}{s^\alpha} \quad \text{for} \quad s^L - s^\alpha \le x \le s^L, \quad \psi(x) = \frac{(s^U + s^\beta) - x}{s^\beta} \quad \text{for} \quad s^U \le x \le s^U + s^\beta.$$

Approximate Operations on Linear T - numbers

A description of basic arithmetic operations with T-numbers is derived by means of the well-known "Extension Principle" [Zadeh73]. Applying this principle, operations between fuzzy objects can be defined by means of classical operations among classical (deterministic) numbers.

For linear T-numbers $\tilde{s} \in F_{TL}(E)$ are suggested approximate operations (sometimes depicted speed operations see[Dubois80]) as follows [Bonissone82].

(1) ***Fuzzy addition*** $\left|\oplus, \tilde{\Sigma}\right|$ for $\tilde{s}, \tilde{r} \in F_{TL}(E)$

$$\tilde{s} \oplus \tilde{r} = \left(s^L; s^U; s^\alpha; s^\beta\right) \oplus \left(r^L; r^U; r^\alpha; r^\beta\right) = \left(s^L + r^L; s^U + r^U; s^\alpha + r^\alpha; s^\beta + r^\beta\right)$$

(2)***Fuzzy subtraction***: $\left(\tilde{-}; \tilde{\tilde{\Sigma}}\right)$ for $\tilde{s}, \tilde{r} \in F_{TL}(E)$,

$$\tilde{s} \tilde{-} \tilde{r} = \left|s^L - r^L; s^U - r^U; s^\alpha + r^\beta; s^\beta + r^\alpha\right|$$

(3) *Approximation of linear T-numbers* ***product*** $\left|\otimes; \tilde{\Pi}\right|$

$$\tilde{s} \otimes \tilde{r} = \left(s^L \cdot r^L; s^U \cdot r^U; s^L \cdot r^\alpha + r^L \cdot s^\alpha - s^\alpha \cdot r^\alpha; s^U \cdot r^\beta + r^U \cdot s^\beta - s^\beta \cdot r^\beta\right) \text{ for } \tilde{s}; \tilde{r} > 0,$$

$$\tilde{s} \otimes \tilde{r} = \left(s^L \cdot r^U; s^U \cdot r^L; r^U \cdot s^\alpha - s^L \cdot r^\beta + s^\alpha \cdot r^\alpha; r^U \cdot s^\beta - s^U \cdot r^\alpha - s^\beta \cdot r^\alpha\right) \text{ for } \tilde{s} < 0; \tilde{r} > 0,$$

$$\tilde{s} \otimes \tilde{r} = \left(s^U \cdot r^U; s^L \cdot r^L; -s^U \cdot r^\beta - r^U \cdot s^\beta - s^\beta \cdot r^\beta; -s^L \cdot r^\alpha - r^L \cdot s^\alpha + s^\alpha \cdot r^\beta\right) \text{ for } \tilde{s}; \tilde{r} < 0.$$

Positive fuzzy set, $\tilde{s} > 0$,is defined, for every $x \in \operatorname{supp} \tilde{s}$, $x > 0$. *Negative fuzzy set* is defined analogously, $\tilde{s} < 0$, for every $x \in \operatorname{supp} \tilde{s}$, $x > 0$,
where $\operatorname{supp} \tilde{s} = \{x \in X; \mu_{\tilde{s}}(x) > 0\}$.

(4) ***Division*** *approximation of linear T-numbers* ($\tilde{\div}$; slash is understood as the fuzzy division operation)

$$\tilde{t} = \left(\frac{s^L}{r^U}; \frac{s^U}{r^L}; \frac{s^L \cdot r^\beta + r^U \cdot s^\alpha}{r^U \cdot (r^U + r^\beta)}; \frac{s^U \cdot r^\alpha + r^L \cdot s^\beta}{r^L \cdot (r^L - r^\alpha)}\right), \quad \text{for } \tilde{s}, \tilde{r} > 0.$$

As for the fuzzy sum and subtraction, it is to full extent an operation on the T-numbers class. This is not complied, however, with the fuzzy product operation, not directed rigorously to the T-numbers with a part-by part linear membership function. Therefore approximation is introduced, the result of which are T-numbers in their final consequence.

Binary fuzzy preference relations of inequality

Three basic conceptions of fuzzy preference relations [Dubois80, Ramík95, Delgado88] are the most important for creation of a binary fuzzy relation. A relation of "epsilon delta $R_{\varepsilon\delta}$ [Ramík95], will be applied and is the most sufficient for the purpose of modelling semantic of various fuzzy relations types on T-numbers in financial models because of good interpretation and computing demandingness extent. This relation is defined for a case of LESS OR EQUAL as follows:

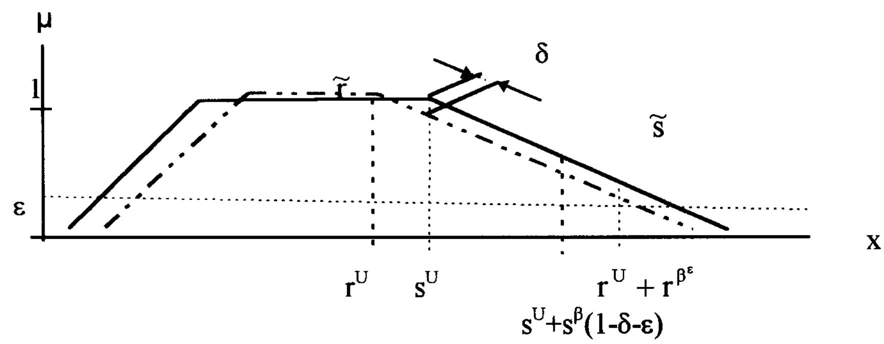
$$\widetilde{RL}_{\varepsilon,\delta} \equiv \tilde{s} \leq^{\widetilde{RL}}_{\varepsilon\delta} \tilde{r} \text{ , if } \mu_{\tilde{s}^{RT}}(x) - \mu_{\tilde{r}^{RT}}(x) \leq \delta \quad \text{for} \quad x \in \left[r^U, r^U + r^{\beta^\varepsilon}\right],$$

$$\text{where} \quad r^{\beta^\varepsilon} = \mu_{\tilde{r}}^{-1}(\varepsilon) - r^U \; ; \; \varepsilon, \delta \in [0;1] .$$

This relation could be transformed according to two linear inequations as follows:

$$s^U \leq r^U + \delta \cdot r^\beta$$
$$s^U + s^\beta \cdot (1 - \delta - \varepsilon) \leq r^U + (1 - \varepsilon) \cdot r^\beta$$

Illustration of $RL_{\varepsilon\delta}$ relation



Objective function

A choice of a method and semantic of a global fuzzy criterion is very important in fuzzy dynamic systems with optimum control. Maximising (shift to the right) of all fuzzy objects points is applied in fuzzy maximisation on T-numbers as one of the approaches. Thus, a following equation applies for linear T-numbers:

$$\widetilde{\max}\ \tilde{z} = \left\{\max\left[z^U; z^\beta\right],\ \min\left[z^\alpha; z^U - z^L\right]\right\}$$

Optimisation vector technique (multicriterial) is applied for solving based on (1) utility function, (2) goal programming, (3) fuzzy programming, (4) interactive methods. The simplest method is utility function and concept possibility value:

$$\widetilde{\max}\ \tilde{z} \equiv \max\ (z_1; z_2; z_3; z_4) = \sum_{i=1}^{4} v_i \cdot z_i = \frac{1}{6}\left(z_1 + 2z_2 + 2z_3 + z_4\right), \text{ where}$$

$z_1 = z^L - z^\alpha$, $z_2 = z^L$, $z_3 = z^U$, $z_4 = z^U + z^\beta$ and v_i entitles weights.

4. Applied fuzzy-scenario optimisation bond portfolio decision problem

For the sake of simplicity and tractability the fuzzyfication of the interest rates is only considered. Further the problem of fuzzy optimisation of bond portfolio is constructed with linear T-numbers, approximate operations and $R_{\varepsilon\delta}$ relation, see chapter 3. The impact of vagueness of interest rates influence the formulae of fuzzy present value and duration as follows:

$$\tilde{p}v = \sum_t cf_t \otimes (1 \oplus \tilde{y}_t)^{-t}, \quad \tilde{d} = p\tilde{v}^{-1} \otimes \sum_t t \otimes cf_t \otimes (1 \oplus \tilde{y}_t)^{-t}, \qquad (1')\ (2')$$

and constraints (3, 4) are modified by this way,

$$\sum_j p\tilde{s}_{j,i,t} \otimes (bh_j - BS_j) \oplus \sum_k p\tilde{b}_{k,i,t} \otimes BB_k \oplus NP_t \overset{RL\varepsilon\delta}{\leq} \sum_j p\tilde{s}_{j,i,t} \otimes BH_{j,i,t} \oplus OP_t \qquad (3')$$

t=1..m, i=1..n,

where $NP_t = Z_t + Z_{t-1}(1+z_{t-1}) - [S_t + S_{t-1}(1+s_{t-1})]$, $OP_t = Z'_t + Z'_{t-1}(1+z'_{t-1}) - [S'_t + S'_{t-1}(1+s'_{t-1})]$,

$$\sum_j \tilde{d}_j \otimes \tilde{p}s_j \otimes (bh_j - BS_j) \oplus \sum_k \tilde{d}_k \otimes \tilde{p}b_k \otimes BB_k \overset{RL\varepsilon\delta}{\leq} \tilde{d}\max, \qquad (4')$$

The above simplified model of the construction of the optimal bond portfolio is illustrated on the verified variant under czech market bonds portfolio. Input data are summarised in table 1 and table 2. For a sake of brevity, the parallel shifts of the interest rates are considered. There is also assumed the duration limit.

The solution, optimal bond portfolio and other cash input ,output, are apparent from tables 1 and 2. The deterministic and fuzzy solutions are compared. It turns out the solution of the fuzzy model is more diversified and less sensitive on irregular development of interest rates. This finding out confirms the fuzzy model might be more effective and advanced means in comparing with deterministic approaches because of the bigger roughness.

Tab. 1 Characteristics of bonds and deterministic and fuzzy portfolio optimal solution

INPUT						**Deterministic solution**	**Fuzzy solution**
BOND	**COUPON**	**Maturity date**	**Price**	**Accrued interest**	**Par value**		
CEZ1	10,90%	27.6.2001	99	817,5	10000	**0**	**100000**
CEZ2	11,06%	27.6.2001	110,35	829,69	10000	**0**	**100000**
CEZ3	11,30%	6.6.2005	101,08	913,42	10000	**100000**	**100000**
CEZ4	14,37%	27.1.2001	100	239,58	10000	**0**	**100000**
CSOB1	11,00%	14.6.2000	99,45	864,72	10000	**100000**	**100000**
CSOB2	11,12%	26.8.1997	99,3	1304,1	20000	**200000**	**200000**
IPB1	11,12%	12.7.1998	98,85	231,77	10000	**100000**	**0**
IPB2	10,90%	17.5.2001	100	393,61	10000	**0**	**100000**
KB1	11,10%	20.5.1999	99,5	835,58	10000	**100000**	**100000**
KB2	11,40%	23.4.1999	100,5	566,83	10000	**0**	**100000**
KB3	12,66%	6.6.2000	101	446,7	10000	**0**	**0**
KB4	12,86%	11.7.1999	100	92,9	10000	**292338**	**0**
Plzeň	11,50%	24.2.2000	99,95	929,58	10000	**0**	**0**

Multisys	14,87%	31.10.1999	89,33	607,4	10000	**0**	**980000**
Nová huť	12,70%	18.11.2003	100,5	455,08	10000	**0**	**0**
OKD	12,50%	18.10.1999	105,9	640,42	10000	**0**	**0**
Setuza	12,20%	20.2.2000	96,9	125,39	10000	**0**	**0**
Vítkovice	13,50%	22.6.2000	100,3	1031,25	10000	**0**	**0**
ZPS Zlín	18,50%	1.3.1998	109,5	6680,55	500000	**0**	**0**

Tab. 2 Interest rates for scenario variants, reinvestment and loan rates

	date	**1.1.1997**	**1.1.1998**	**1.1.1999**	**1.1.2000**	**1.1.2001**
INPUT	**scenario 1**	0,11	0,11	0,11	0,11	0,11
	scenario 2	0,11	0,12	0,13	0,13	0,14
	scenario 3	0,11	0,12	0,11	0,10	0,10
	scenario 4	0,11	0,10	0,12	0,12	0,12
	salfa	0,001	0,001	0,001	0,001	0,001
	sbeta	0,002	0,002	0,002	0,002	0,002
	liabilities		20000,00	30000,00	40000,00	50000,00
	reinvest. rate	0,07	0,07	0,07	0,07	0,07
	lending rate	0,11	0,11	0,11	0,11	0,11
FUZZY SOLUTION	**reinvestment**	499178	490967	604421	1405816	1160083
	loan		0	0	32722	0
DETERMIN. SOLUTION	**reinvestment**	575904375	616154863	659245126	705555166	754688075
	loan		0	0	32722	0

Conclusion

This paper described fuzzy scenario optimisation model of bond portfolio decision-making. Attention was devoted to possibility of application of fuzzy programming technique to modelling of development of yield curves. This type of the troubles is a typical also for a behaviour of the interest rates in the developed and emerging markets. In spite of that it is difficult to predict the shape of the yield curves, portfolio managers may generate a number of possible interest rate scenarios applying fuzzy apparatus. As was introduced on illustrative example, fuzzy-scenario optimisation model is one way how to include uncertainty assumption of the yield curves shapes in bond portfolio optimisation modelling.

A traditional conception of complex dynamic systems is based on the idea the fuzzy approach is additionally inserted into a model. However, the opposite is true, a deterministic model being something like an inter-member and special fuzzy model case. It was verified, the fuzzy model could be considered to be simultaneously a sensitivity analysis of a decision-making model. From this point of view, the described fuzzy modelling method is a common sensitivity analysis comparising elements of roughness. Thus a fuzzy model can be considered to be a generalised approach towards modelling bond portfolio optimisation decision problem in a financial sphere.

Summary

This paper describes a comparison of the deterministic and fuzzy optimisation models behaviour. The fuzzy-scenario methodology of bond portfolio optimisation decision-making is used. Attention is devoted to possibility of application of a fuzzy apparatus to modelling of dynamic terms structure uncertainty and cash-flows. Influence of developing changes of interest rates on value of bonds, deterministic and fuzzy problem of bond portfolio optimisation is formulated, fuzzy methodology (fuzzy set of T-number, approximate operations, $R_{\varepsilon\delta}$ relation, fuzzy objective function) is applied. Illustrative example is described. It was verified, the fuzzy model could be considered to be simultaneously a sensitivity analysis of a decision-making model. From this point of view, the described fuzzy modelling method is a common sensitivity analysis comparing elements of roughness. Thus a fuzzy model can be considered to be a generalised approach towards modelling of bond portfolio optimisation decision-making problem in a financial sphere

Keywords: bond portfolio optimisation decision technique, fuzzy programming, dedicated, immunisation, scenario methodology, duration, present value, T-fuzzy number, approximate fuzzy operations, $R_{\varepsilon\delta}$ binary fuzzy relation.

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